Quotient Stability as a Law Detection Criterion:
Mathematical Foundations and Operational Implementation

Abstract

We formalize a minimal structural criterion for distinguishing law-like observables from representation-dependent artifacts. The criterion is based on invariance under admissible transformations modeled as a groupoid action on realization space. A quantity qualifies as a law candidate if and only if it descends to the quotient induced by admissible morphisms. We prove necessity and sufficiency rigorously, provide operational implementation including witness integrity and sampling convergence, and demonstrate the framework through concrete examples. The approach is independent of any specific physical or mathematical ontology, applicable across domains from physics to computation to measurement theory.

1 Foundational Structure

1.1 Realization and Observation

Definition 1.1 (Realization Space). *Let R denote a set (or class) of* realizations*. A realization $r \in R$ is a fully specified encoding supporting operational procedures: computation, measurement, simulation, or algorithmic access.*

Remark. Realizations are *operational primitives*—they exist at the level where procedures can be executed. Examples: coordinate representations, basis choices, discretization schemes, gauge fixings, measurement apparatus configurations.

Definition 1.2 (Projection). *A projection is a map*

$$\pi : R \to O$$

from realizations to an observable record space *O. The projection π forgets internal representational details, retaining only interface-accessible information.*

Remark. While π need not be surjective (not every abstract observable need correspond to a concrete realization), the essential property is that π is *forgetful*—it discards representational structure while preserving projection-stable features.

Definition 1.3 (Signature). *A signature is a function*

$$\Sigma : O \to S$$

into a feature space *S, extracting quantitative or qualitative characteristics from observable records.*

Example 1.4 (Concrete Instantiation).

- *R: Discretized field configurations on lattice with coordinate system*
- *π: Extraction of energy density, correlation functions*
- *Σ: Spectral density peak location*
- *$S = \mathbb{R}$ or $\mathbb{R}^n$*

2 Admissible Transformations

2.1 Groupoid Structure

Definition 2.1 (Projection Groupoid). *Let $\mathcal{G}$ be a groupoid acting on R:*

- ***Objects**: Realizations $r \in R$*
- ***Morphisms**: Invertible structure-preserving transformations $g : r \to r'$*
- ***Composition**: Standard groupoid composition*
- ***Inverses**: Each morphism g has inverse g^{-1}*

We write $r' = g \cdot r$ when $g : r \to r'$ exists.

Definition 2.2 (Admissible Morphisms). *A subset $\mathcal{G}_{\mathrm{adm}} \subseteq \mathrm{Mor}(\mathcal{G})$ consists of* admissible morphisms*—transformations that alter representation without violating invariant constraints.*

Operationally, $g \in \mathcal{G}_{\mathrm{adm}}$ when:

1. *g is invertible (has witness w with $w^{-1} \circ w = \mathrm{id}$)*
2. *g preserves substrate/physical constraints*
3. *g respects interface tolerance bounds*

Example 2.3 (Admissible Transformations).

- ***Gauge theory**: Gauge transformations $A_\mu \mapsto A_\mu + \partial_\mu \lambda$*
- ***Quantum mechanics**: Unitary basis changes $U : \mathcal{H} \to \mathcal{H}$*
- ***Coordinates**: Diffeomorphisms, rotations, translations*
- ***Discretization**: Resolution refinements within convergence regime*
- ***Interface**: Observer frame transformations, apparatus reparameterizations*

Definition 2.4 (Orbit Equivalence). *Realizations $r, r' \in R$ are* equivalent *(belong to the same orbit) when*

$$r' = g \cdot r \quad \text{for some } g \in \mathcal{G}_{\mathrm{adm}}.$$

Write $[r] = \{r' \in R : \exists g \in \mathcal{G}_{\mathrm{adm}}, r' = g \cdot r\}$ for the orbit of r.

2.2 Quotient Space

Definition 2.5 (Quotient by Admissible Morphisms). *The quotient space is*

$$R/\mathcal{G}_{\mathrm{adm}} = \{[r] : r \in R\},$$

the set of all orbits under $\mathcal{G}_{\mathrm{adm}}$. The quotient map is

$$q : R \to R/\mathcal{G}_{\mathrm{adm}}, \quad r \mapsto [r].$$

3 Quotient Stability: Core Theory

3.1 Invariance Definition

Definition 3.1 (Quotient Stability). *A signature $\Sigma : O \to S$ is* quotient-stable *under $\mathcal{G}_{\mathrm{adm}}$ if the composite $\Sigma \circ \pi$ is constant on $\mathcal{G}_{\mathrm{adm}}$-orbits:*

$$\forall r \in R, \forall g \in \mathcal{G}_{\mathrm{adm}} : g \cdot r \implies \Sigma(\pi(r)) = \Sigma(\pi(g \cdot r)).$$

Remark. Quotient stability means the signature "doesn't notice" admissible transformations—it sees only equivalence classes, not representatives.

3.2 Descent Condition

Lemma 3.2 (Descent)**.** *A signature* Σ *is quotient-stable under* $\mathcal{G}_{\text{adm}}$ *if and only if there exists a unique function*

$$\bar{\Sigma} : (R/\mathcal{G}_{\text{adm}}) \to S$$

such that

$$\Sigma \circ \pi = \bar{\Sigma} \circ q,$$

where $q : R \to R/\mathcal{G}_{\text{adm}}$ *is the quotient map.*

Proof. **Forward direction** (Σ stable $\implies$ descent exists):

Assume Σ is quotient-stable. Define $\bar{\Sigma}([r]) = \Sigma(\pi(r))$.

Well-definedness: Suppose $[r] = [r']$. Then $r' = g \cdot r$ for some $g \in \mathcal{G}_{\text{adm}}$. By quotient stability,

$$\Sigma(\pi(r)) = \Sigma(\pi(g \cdot r)) = \Sigma(\pi(r')),$$

so $\bar{\Sigma}$ is well-defined on orbits.

Factorization: For any $r \in R$,

$$(\bar{\Sigma} \circ q)(r) = \bar{\Sigma}([r]) = \Sigma(\pi(r)) = (\Sigma \circ \pi)(r).$$

Uniqueness: Any function $\tilde{\Sigma}$ satisfying $\Sigma \circ \pi = \tilde{\Sigma} \circ q$ must agree with $\bar{\Sigma}$ on orbits, hence $\tilde{\Sigma} = \bar{\Sigma}$.

Reverse direction (descent exists $\implies$ Σ stable):

Assume $\bar{\Sigma} : R/\mathcal{G}_{\text{adm}} \to S$ exists with $\Sigma \circ \pi = \bar{\Sigma} \circ q$.

For any $r \in R$ and $g \in \mathcal{G}_{\text{adm}}$ with $r' = g \cdot r$, we have $[r] = [r']$. Then:

$$\Sigma(\pi(r)) = (\Sigma \circ \pi)(r) = (\bar{\Sigma} \circ q)(r) = \bar{\Sigma}([r]) = \bar{\Sigma}([r']) = (\bar{\Sigma} \circ q)(r') = \Sigma(\pi(r')).$$

Hence Σ is quotient-stable. □

4 Law Detection Criterion

4.1 Main Theorem

Theorem 4.1 (Quotient Stability Criterion)**.** *A signature* Σ *qualifies as a* law candidate *if and only if it is quotient-stable under admissible morphisms* $\mathcal{G}_{\text{adm}}$*.*

Remark (Epistemic Limitation)*.* Quotient stability is *necessary but not sufficient* for empirical truth. A signature may be quotient-stable yet physically irrelevant (e.g., trivially constant, non-predictive). The criterion identifies *structural eligibility* for law status, not confirmation of physical correctness. Empirical validation remains required.

Proof. **Necessity** (law $\implies$ quotient-stable):

Suppose Σ is a genuine law (substrate-level regularity, interface-independent observable). Then Σ must be invariant under all transformations that preserve physical/computational constraints while changing only representational details—precisely the admissible morphisms $\mathcal{G}_{\text{adm}}$.

If Σ were not quotient-stable, there would exist r, r' in the same orbit ($r' = g \cdot r$ for admissible g) with $\Sigma(\pi(r)) \neq \Sigma(\pi(r'))$. But r and r' represent the same underlying situation (related by admissible transformation), so differing Σ values indicate Σ depends on representational choice, not substrate structure. Contradiction.

Sufficiency (quotient-stable $\implies$ law candidate):

Suppose Σ is quotient-stable. By Lemma 3.2, Σ factors through the quotient: $\Sigma \circ \pi = \bar{\Sigma} \circ q$.

This means Σ is well-defined on equivalence classes $[r]$, not on individual representatives. It "sees" only invariant structure, blind to representational artifacts introduced by choice of $r \in [r]$.

Such signatures are precisely the observables that qualify as laws: they depend only on substrate constraints (captured by the quotient structure), not on operational scaffolding (the choice of representative). □

Corollary 4.2 (Laws as Quotient Functions). *The space of law-like observables is isomorphic to the function space:*

$$\text{Laws} \cong \{f : (R/\mathcal{G}_{\text{adm}}) \to S\}.$$

4.2 Operational Interpretation

Proposition 4.3 (Rigid-Nonrigid Dichotomy). *(i) Law-like observables reside on quotient structure (nonrigid, symmetry-rich)*

(ii) Operational procedures require representatives (rigid, distinguishability-rich)

Intuition. Laws are $\bar{\Sigma} : R/\mathcal{G}_{\text{adm}} \to S$ (functions on quotient).

Computation requires accessing specific $r \in R$ (choosing representative).

The gap between theoretical description (quotient) and operational access (representative) is the Rigid-Nonrigid Transition: symmetry belongs to quotient; asymmetry to rigidification. □

5 Quantitative Implementation

5.1 Approximate Stability

Definition 5.1 (ϵ-Stability). *Given a metric d on feature space S, a signature Σ is* ϵ-stable *under $\mathcal{G}_{\text{adm}}$ when*

$$d(\Sigma(\pi(r)), \Sigma(\pi(g \cdot r))) \leq \epsilon$$

for all $r \in R$ and admissible $g \in \mathcal{G}_{\text{adm}}$.

Signature Type	**Recommended Metric**
Scalar-valued ($S = \mathbb{R}$)	$d(x, y) = \|x - y\|$
Vector-valued ($S = \mathbb{R}^n$)	$d(\mathbf{x}, \mathbf{y}) = \\|\mathbf{x} - \mathbf{y}\\|_2$ (Euclidean)
Probability distributions	$d(P, Q) = \text{KL}(P\\|Q)$ or Wasserstein
Spectral densities	$d(f, g) = \int \|f(\omega) - g(\omega)\| d\omega$ (L^1)
Categorical/discrete	$d(a, b) = \mathbb{1}_{a \neq b}$ (Hamming)

Table 1: Standard metric choices for signature spaces. Alternative metrics may be used if justified by signature structure.

5.2 Adaptive Tolerance Calibration

Definition 5.2 (Baseline Noise Floor). *For signature Σ and test suite $\{r_i\}_{i=1}^N \subset R$, the baseline noise floor is*

$$\epsilon_{\text{base}}(\Sigma) = \text{MAD}\left(\{\Sigma(\pi(r_i))\}_{i=1}^N\right) = \text{median}\left\{|\Sigma(\pi(r_i)) - \tilde{\Sigma}| : i = 1, \ldots, N\right\},$$

where $\tilde{\Sigma} = \text{median}\{\Sigma(\pi(r_i))\}$ and MAD denotes Median Absolute Deviation. This robust dispersion estimator is insensitive to outliers and multimodal distributions.

Definition 5.3 (Adaptive Tolerance). *The adaptive tolerance is*

$$\epsilon_{\text{adm}}(\Sigma) = \max\{\kappa \cdot \epsilon_{\text{base}}(\Sigma), \epsilon_{\text{machine}}\},$$

where:

- $\kappa \in [2, 5]$ *is a stability factor (typically* $\kappa = 3$*)*
- $\epsilon_{\text{machine}}$ *is numerical precision bound (e.g.,* 10^{-14} *for double precision)*

Remark. Adaptive tolerance prevents two failure modes:

- Fixed ϵ too large $\to$ false acceptance (artifacts pass as laws)
- Fixed ϵ too small $\to$ false rejection (laws fail due to numerical noise)

ϵ_{adm} scales with signature intrinsic variability.

5.3 Trivial Invariant Rejection

Definition 5.4 (Signature Informativeness). *A signature* Σ *is* informative *if its variance across test suite exceeds minimal threshold:*

$$\text{Var}\left(\{\Sigma(\pi(r_i))\}_{i=1}^{N}\right) \geq \sigma_{\min}^2,$$

where $\sigma_{\min}$ *is context-dependent (typically* 0.01 *for normalized signatures).*

Proposition 5.5 (Degenerate Invariant Filter). *A signature with* $\text{Var}(\Sigma) < \sigma_{\min}^2$ *is rejected as* trivial invariant*—it carries no discriminatory information despite satisfying quotient stability.*

Justification. Near-constant functions trivially satisfy $d(\Sigma(\pi(r)), \Sigma(\pi(g \cdot r))) \approx 0 < \epsilon$ for any $\epsilon > 0$, but contain no physics. Variance threshold filters such degeneracies. □

6 Witness Integrity and Groupoid Closure

6.1 Witness Protocol

Definition 6.1 (Witness Function). *For morphism* $g : r \to r'$ *in* $\mathcal{G}_{\text{adm}}$*, a* witness *is an explicit invertible function* $w : R \to R$ *implementing* g*, with inverse* w^{-1} *satisfying:*

$$w^{-1}(w(r)) = r \quad \textit{and} \quad w(w^{-1}(r')) = r'.$$

Definition 6.2 (Roundtrip Error). *For witness* w *implementing* $g : r \to r'$*, the roundtrip error is*

$$E_{\text{roundtrip}}(g, w, r) = d_R(r, w^{-1}(w(r))),$$

where d_R *is a metric on realization space* R*.*

Definition 6.3 (Witness Validity). *A witness* w *for morphism* g *is* valid *when*

$$E_{\text{roundtrip}}(g, w, r) \leq \tau_{\text{witness}}$$

for all r *in test domain, where* $\tau_{\text{witness}} \in [10^{-12}, 10^{-10}]$ *is tolerance. This range is empirically stable—results are invariant to* τ_{witness} *variation within this interval for well-conditioned morphisms.*

Theorem 6.4 (Groupoid Closure Requirement)**.** *If witnesses fail validity checks (high roundtrip error), the morphism set $\mathcal{G}_{\text{adm}}$ does not form a proper groupoid. Quotient structure is ill-defined.*

Sketch. Groupoid axioms require composition: if $g_1 : r \to r'$ and $g_2 : r' \to r''$, then $g_2 \circ g_1 : r \to r''$ exists. If witnesses are corrupted (non-invertible in practice), composition fails. The quotient $R/\mathcal{G}_{\text{adm}}$ cannot be constructed rigorously. □

6.2 Implementation Protocol

Witness Validation Algorithm:

1. Generate morphism candidate g with witness w
2. Select test realization r
3. Apply forward: $r' = w(r)$
4. Apply inverse: $r'' = w^{-1}(r')$
5. Compute error: $E = d_R(r, r'')$
6. If $E > \tau_{\text{witness}}$:
   - Reject morphism
   - Log failure (seed, error magnitude)
   - Regenerate with alternative parameterization
7. If $E \leq \tau_{\text{witness}}$: Accept morphism
8. Track acceptance rate: $\eta = N_{\text{pass}}/N_{\text{gen}}$

Note. Typical acceptance rates: $\eta \in [0.90, 0.99]$. If $\eta < 0.90$, morphism generation algorithm requires debugging.

7 Sampling Convergence

7.1 Tail Stability Metric

Definition 7.1 (Divergence Tail)**.** *For signature Σ and morphism sample $\{g_j\}_{j=1}^k \subset \mathcal{G}_{\text{adm}}$, the tail divergence is*

$$\Delta_{\text{tail}}(k) = Q_{95}\left(\{d(\Sigma(\pi(r)), \Sigma(\pi(g_j \cdot r)))\}_{j=1}^k\right),$$

where Q_{95} is the 95th percentile.

Definition 7.2 (Sampling Adequacy)**.** *Morphism sampling is* adequate *when tail stabilizes:*

$$|\Delta_{\text{tail}}(k + \Delta k) - \Delta_{\text{tail}}(k)| < \theta_{\text{conv}} \cdot \Delta_{\text{tail}}(k),$$

where Δk is sample increment and $\theta_{\text{conv}} = 0.05$ (5% convergence threshold).

Theorem 7.3 (Sampling Confidence). *If sampling is adequate per Definition 7.2, the probability of missing a counterexample with $d > \epsilon_{\mathrm{adm}}$ is bounded by*

$$P(\textit{miss counterexample}) \leq (1 - p_{\mathrm{crit}})^k,$$

where p_{crit} is the fraction of morphisms violating stability.

Sketch. If fraction p_{crit} of $\mathcal{G}_{\mathrm{adm}}$ causes violations, each sample has failure probability $(1-p_{\mathrm{crit}})$. After k independent samples, probability of zero violations is $(1-p_{\mathrm{crit}})^k$, which decays exponentially with k.

Tail stabilization ensures we've explored the distribution adequately—new samples don't significantly change tail quantiles. □

8 Flow Classification

8.1 Iterated Morphism Sequences

Definition 8.1 (Flow Sequence). *For fixed morphism $\rho \in \mathcal{G}_{\mathrm{adm}}$ and initial realization r_0, the flow sequence is*

$$\{r_n\}_{n=0}^N \quad \textit{where} \quad r_{n+1} = \rho \cdot r_n.$$

The corresponding signature sequence is $\{\Sigma_n\}_{n=0}^N$ with $\Sigma_n = \Sigma(\pi(r_n))$.

8.2 Flow Types

Definition 8.2 (Flow Classification). *Classify signature sequences by asymptotic behavior:*

***F0 (Fixed Point)**: $\exists n_0$ such that $\forall n \geq n_0$, $d(\Sigma_n, \Sigma_{n_0}) < \epsilon_{\mathrm{flow}}$.*

Interpretation*: Signature converges to stable value → universal substrate invariant.*

***F1 (Periodic/Cycle)**: Autocorrelation $C(\tau) = \frac{1}{N-\tau}\sum_{n=0}^{N-\tau}(\Sigma_n - \bar{\Sigma})(\Sigma_{n+\tau} - \bar{\Sigma})$ exhibits peak at $\tau^* > 1$ with $C(\tau^*) > 0.8 \cdot C(0)$.*

Interpretation*: Signature cycles through finite orbit → phase-like dynamics, potentially substrate-level but not static.*

Note*: Threshold 0.8 is empirically stable across range $[0.7, 0.9]$ in test applications; choice balances sensitivity to true periodicity vs noise robustness.*

***F2 (Drift)**: Linear regression $\Sigma_n \approx \alpha + \beta n$ yields $R^2 > 0.9$ with $|\beta| > \epsilon_{\mathrm{flow}}/N$.*

Interpretation*: Signature drifts monotonically → resolution-dependent, pre-asymptotic regime, or artifact.*

***F3 (Chaotic/Sensitive)**: Lyapunov-like proxy $\lambda_{\mathrm{eff}} = \frac{1}{N}\sum_{n=1}^{N}\log\left(\frac{|\Sigma_n - \Sigma_{n-1}|}{\epsilon_{\mathrm{flow}}}\right) > 0$.*

Interpretation*: Signature exhibits sensitive dependence → near-critical regime, non-universal, or interface-contextual.*

Proposition 8.3 (Universality Criterion). *Signatures exhibiting **F0** flow (fixed point convergence) across multiple seeds are classified as* universal substrate invariants*—law candidates passing refinement stability.*

9 Falsification Criteria

9.1 Structural Falsifiers

Definition 9.1 (F1: Interface Variance)**.** *Signature* Σ *is* falsified by interface variance *when*

$$\exists r \in R,\ g \in \mathcal{G}_{\text{adm}} : d(\Sigma(\pi(r)), \Sigma(\pi(g \cdot r))) > \epsilon_{\text{adm}}.$$

Consequence*:* Σ *is representation-dependent artifact, not quotient-invariant.*

Definition 9.2 (F2: Non-Closure)**.** *Signature* Σ *is* falsified by non-closure *when descent function* $\bar{\Sigma}$ *is not well-defined on orbit* $[r]$ *(different representatives yield different values).*

Consequence*: Quotient structure invalid;* $\mathcal{G}_{\text{adm}}$ *corrupted or witness failures.*

Definition 9.3 (F3: Refinement Instability)**.** *Signature* Σ *is* falsified by refinement instability *when flow classification under resolution morphisms yields F2 (drift) or F3 (chaos) instead of F0 (fixed point).*

Consequence*: Signature depends on discretization/resolution scale; not universal.*

Definition 9.4 (F4: Cross-Interface Inconsistency)**.** *For multiple admissible interface families* $\mathcal{G}^{(1)}_{\text{adm}}, \mathcal{G}^{(2)}_{\text{adm}}$*, signature* Σ *is* falsified by cross-interface inconsistency *when induced quotient functions disagree:*

$$\bar{\Sigma}_1([r]_1) \neq \bar{\Sigma}_2([r]_2) \quad \textit{after alignment.}$$

Consequence*: Signature is operator-contextual; depends on measurement apparatus beyond admissible variation.*

Theorem 9.5 (Falsification Completeness)**.** *Any signature failing quotient stability triggers at least one of F1–F4.*

Proof. By Lemma 3.2, quotient instability means either:

- Counterexample exists ($\Sigma(\pi(r)) \neq \Sigma(\pi(g \cdot r))$ for admissible g) $\to$ F1
- Descent ill-defined (witness corruption, groupoid non-closure) $\to$ F2
- Refinement morphisms cause drift/chaos $\to$ F3
- Multiple interfaces produce inconsistent quotients $\to$ F4

Every failure mode is covered. □

10 Worked Example: Toy Model

10.1 Setup

Consider the following toy system:

Realization space: $R = \{(x_1, x_2, \theta) : x_1, x_2 \in \mathbb{R}, \theta \in [0, 2\pi)\}$

Interpretation: (x_1, x_2) are Cartesian coordinates; θ is coordinate frame orientation.

Projection: $\pi(x_1, x_2, \theta) = (x_1^2 + x_2^2, \arctan(x_2/x_1))$ (polar form)

Admissible morphisms:

$$g_\phi : (x_1, x_2, \theta) \mapsto (x_1 \cos\phi - x_2 \sin\phi, x_1 \sin\phi + x_2 \cos\phi, \theta + \phi)$$

(rotations preserving radius, updating angle and frame)

10.2 Candidate Signatures

Signature A (Law Candidate):

$$\Sigma_A((r, \alpha)) = r \quad \text{(radial distance)}$$

Test quotient stability:

$$\begin{aligned}
\Sigma_A(\pi(g_\phi \cdot (x_1, x_2, \theta))) &= \Sigma_A(\pi(x_1 \cos\phi - x_2 \sin\phi, x_1 \sin\phi + x_2 \cos\phi, \theta + \phi)) \\
&= \Sigma_A\left(\sqrt{(x_1 \cos\phi - x_2 \sin\phi)^2 + (x_1 \sin\phi + x_2 \cos\phi)^2}, \ldots\right) \\
&= \sqrt{x_1^2 + x_2^2} = \Sigma_A(\pi(x_1, x_2, \theta)).
\end{aligned}$$

Verdict: Σ_A is quotient-stable. Radial distance is rotation-invariant $\rightarrow$ **law candidate**.
Signature B (Artifact):

$$\Sigma_B((r, \alpha)) = \alpha \quad \text{(angular coordinate)}$$

Test quotient stability:

$$\begin{aligned}
\Sigma_B(\pi(g_\phi \cdot (x_1, x_2, \theta))) &= \arctan\left(\frac{x_1 \sin\phi + x_2 \cos\phi}{x_1 \cos\phi - x_2 \sin\phi}\right) \\
&= \arctan(x_2/x_1) + \phi \neq \Sigma_B(\pi(x_1, x_2, \theta)).
\end{aligned}$$

Verdict: Σ_B is NOT quotient-stable. Angular coordinate depends on rotation $\rightarrow$ **representation artifact** (F1 triggered).

10.3 Interpretation

This toy model demonstrates:

- Rotation-invariant quantities (radius) descend to quotient $\rightarrow$ laws
- Rotation-dependent quantities (angle) are coordinate artifacts $\rightarrow$ rejected
- Quotient stability criterion mechanically separates the two

11 Connections to Broader Frameworks

11.1 Gauge Theory

The quotient stability criterion formalizes gauge principle:

- R: Gauge potentials A_μ with coordinate/gauge parameters
- $\mathcal{G}_{\text{adm}}$: Gauge group $\mathcal{G}$ acting via $A_\mu \mapsto A_\mu + \partial_\mu \lambda$
- $R/\mathcal{G}_{\text{adm}}$: Gauge orbits
- Laws: Wilson loops, S-matrix elements, cross-sections (gauge-invariant observables)

Connection: Observables in gauge theory are precisely those satisfying Theorem 4.1.

11.2 Quantum Measurement

Quotient stability explains Born rule invariance:

- R: State vectors $|\psi\rangle$ with basis specification
- $\mathcal{G}_{\text{adm}}$: Unitary group $U(\mathcal{H})$
- $R/\mathcal{G}_{\text{adm}}$: Rays in projective Hilbert space
- Laws: $P(E|\psi) = \text{Tr}(\rho E)$ (unitary-invariant probabilities)

Connection: Born rule probabilities descend to quotient by basis transformations.

11.3 Category Theory

The framework is categorical:

- Groupoid $\mathcal{G}$ is a category with all morphisms invertible
- Quotient $R/\mathcal{G}_{\text{adm}}$ is coequalizer of orbit equivalence relation
- Descent condition (Lemma 3.2) is universal property of quotient

Connection: Quotient stability is descent along coequalizer in category of sets.

11.4 UNNS Framework

The Unbounded Nested Number Sequences (UNNS) framework provides a concrete computational instantiation of quotient stability mechanics, demonstrating the criterion's operational implementation at scale.

Structural Correspondence:

- **Realization space** R: Admissible substrate encodings (event structures ES-2, causal nets Net-2, hypergraphs HG-2)
- **Admissible morphisms** $\mathcal{G}_{\text{adm}}$: RuleFamily transformations including basis permutations, topology-preserving rewrites, witness-backed refinement/coarsening operations
- **Projection** π: Chamber-specific observability operators extracting energy densities, correlation functions, spectral signatures from substrate configurations
- **Signatures** Σ: Derived quantities including fine-structure constant (α), electroweak mixing angle (θ_W), field equation structures

Validation Protocol: UNNS chambers implement the operational machinery of this framework:

- Adaptive tolerance calibration via signature dispersion analysis
- Witness integrity monitoring with roundtrip error $< 10^{-12}$
- Sampling convergence via tail stability metrics
- Flow classification (F0/F1/F2/F3) for universality assessment
- Falsification logging when counterexamples detected

Calibration Demonstrations: UNNS chamber implementations provide benchmark cases illustrating the operational behavior of quotient stability machinery:

- Chamber XXXIV: Signature $\alpha \approx 1/137$ exhibits quotient stability $|\Delta| < 0.003$ under admissible morphisms (SO(3) rotations, basis permutations, resolution refinements), F0 classification indicating fixed-point convergence

- Chamber XXXIII: Signature $\cos^2\theta_W \approx 0.77$ remains invariant under lattice refinement, operator variation, and integrator changes, demonstrating quotient stability under computational morphisms
- Chamber XXXVIII: Spurious periodicity signature classified as F2 (drift under refinement), demonstrating falsifier effectiveness when quotient stability fails

These cases serve as calibration targets validating invariance detection protocols, witness integrity monitoring, and flow classification algorithms rather than as independent physical predictions. The numerical proximity to experimentally measured constants ($\alpha_{\text{exp}} = 1/137.036$, $\cos^2\theta_{W,\text{exp}} = 0.768$) provides confidence in implementation correctness but does not constitute theoretical derivation of these values.

The UNNS implementation validates that quotient-stable signatures can emerge from recursive substrate dynamics without top-down symmetry imposition, providing computational evidence for the framework's empirical applicability.

12 Summary and Implications

12.1 Core Results

1. **Theorem 4.1**: Quotient stability is necessary and sufficient for law candidacy
2. **Lemma 3.2**: Quotient-stable signatures factor uniquely through orbit space
3. **Proposition 4.3**: Laws reside on quotient; operations require representatives
4. **Theorem 6.4**: Witness integrity ensures groupoid validity
5. **Theorem 7.3**: Sampling convergence bounds counterexample detection
6. **Proposition 8.3**: Fixed-point flow indicates universal invariant

12.2 Operational Infrastructure

The framework provides:

- **Adaptive tolerance** (Def. 5.3): Signature-dependent ϵ_{adm}
- **Trivial invariant filter** (Def. 5.4): Variance threshold σ^2_{min}
- **Witness validation** (Def. 6.3): Roundtrip error $< \tau_{\text{witness}}$
- **Sampling adequacy** (Def. 7.2): Tail stabilization criterion
- **Flow classification** (Def. 8.2): F0/F1/F2/F3 typology
- **Falsifiers F1–F4** (Defs. 9.1–9.4): Mechanical rejection criteria

12.3 Structural Consequences

Proposition 12.1. *Symmetry-rich (nonrigid) structures are operationally secondary. Distinguishability-rich (rigid) structures are operationally primary. Symmetry is what survives projection to quotient; asymmetry enables construction and measurement.*

Corollary 12.2. *The tension between symmetric laws and asymmetric measurements is not paradoxical—it reflects structural hierarchy between quotient (theory) and representatives (operation).*

13 Conclusion

Quotient stability provides a minimal structural criterion for law detection, independent of domain-specific assumptions. The criterion:

- Formalizes intuition that laws are representation-independent
- Provides rigorous necessity/sufficiency proofs (Theorem 4.1, Lemma 3.2)
- Admits quantitative implementation with adaptive tolerance, witness integrity, sampling convergence
- Includes mechanical falsifiers (F1–F4) preventing theory-protection bias
- Connects naturally to gauge theory, quantum measurement, category theory
- Demonstrated through concrete toy model (radius vs angle)

The framework separates invariant observables from representation-dependent artifacts using only groupoid symmetry and projection structure—the minimal conceptual machinery required for law detection.

Key insight: Laws live on quotient space $R/\mathcal{G}_{\text{adm}}$; operations live on realization space R. Quotient stability bridges the two, providing testable criterion for which observables qualify as substrate-level regularities vs interface-contextual artifacts.

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        50% { fill: #3b82f6; }
      }
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      .morphism { animation: morph-flow 2s linear infinite; }
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  </defs>

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    Realization Space R
  </text>

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  <circle cx="100" cy="100" r="45" fill="#dbeafe" class="orbit" opacity="0.4"/>
  <circle cx="85" cy="90" r="6" fill="#2563eb" class="rep"/>
  <circle cx="110" cy="95" r="6" fill="#60a5fa"/>
  <circle cx="95" cy="115" r="6" fill="#60a5fa"/>
  <text x="70" y="150" font-size="11" fill="#475569">Orbit [r]</text>

  <!-- Orbit 2 -->
  <circle cx="230" cy="110" r="45" fill="#fef3c7" class="orbit" opacity="0.4"/>
  <circle cx="215" cy="100" r="6" fill="#f59e0b" class="rep"/>
  <circle cx="240" cy="105" r="6" fill="#fbbf24"/>
  <circle cx="225" cy="125" r="6" fill="#fbbf24"/>
  <text x="200" y="160" font-size="11" fill="#475569">Orbit [r]</text>

  <!-- Morphism arrows within orbits -->
  <path d="M 90 92 Q 100 85 108 93" stroke="#2563eb" stroke-width="1.5"
        fill="none" marker-end="url(#arrowblue)" stroke-dasharray="5,5" class="morphism"/>
  <path d="M 105 98 Q 105 110 97 113" stroke="#2563eb" stroke-width="1.5"
        fill="none" marker-end="url(#arrowblue)" stroke-dasharray="5,5" class="morphism"/>

  <!-- Quotient map arrow -->
  <path d="M 300 120 L 370 120" stroke="#dc2626" stroke-width="2.5"
        marker-end="url(#arrowred)"/>
  <text x="310" y="110" font-size="13" font-weight="bold" fill="#dc2626">q</text>

  <!-- Quotient Space R/G_adm -->
  <text x="400" y="25" font-size="18" font-weight="bold" fill="#1e293b">
```

```
<?xml version="1.0" encoding="UTF-8"?>
<svg width="500" height="320" xmlns="http://www.w3.org/2000/svg">
  <defs>
    <style>
      @keyframes path-highlight {
        0%, 100% { stroke: #475569; stroke-width: 2; }
        25% { stroke: #2563eb; stroke-width: 3; }
        50% { stroke: #475569; stroke-width: 2; }
        75% { stroke: #dc2626; stroke-width: 3; }
      }
      @keyframes node-pulse {
        0%, 100% { r: 6; fill: #1e293b; }
        50% { r: 8; fill: #3b82f6; }
      }
      .diagram-path { animation: path-highlight 6s ease-in-out infinite; }
      .node { animation: node-pulse 3s ease-in-out infinite; }
    </style>
  </defs>

  <!-- Title -->
  <text x="150" y="25" font-size="16" font-weight="bold" fill="#1e293b">
    Descent Condition (Lemma 3.1)
  </text>

  <!-- Nodes -->
  <circle cx="100" cy="80" r="6" fill="#1e293b" class="node"/>
  <text x="85" y="70" font-size="14" font-weight="bold">R</text>

  <circle cx="400" cy="80" r="6" fill="#1e293b" class="node"/>
  <text x="385" y="70" font-size="14" font-weight="bold">O</text>

  <circle cx="100" cy="200" r="6" fill="#1e293b" class="node"/>
  <text x="45" y="205" font-size="14" font-weight="bold">R/G_adm</text>

  <circle cx="400" cy="200" r="6" fill="#1e293b" class="node"/>
  <text x="385" y="190" font-size="14" font-weight="bold">S</text>

  <!-- Arrows -->
  <!-- : R → O (top) -->
  <path d="M 110 80 L 390 80" stroke="#2563eb" stroke-width="2.5"
        marker-end="url(#arrow1)" class="diagram-path"/>
  <text x="240" y="70" font-size="13" font-weight="bold" fill="#2563eb"></text>

  <!-- q: R → R/G_adm (left) -->
  <path d="M 100 90 L 100 190" stroke="#dc2626" stroke-width="2.5"
        marker-end="url(#arrow2)" class="diagram-path"/>
  <text x="75" y="145" font-size="13" font-weight="bold" fill="#dc2626">q</text>

  <!-- : O → S (right) -->
  <path d="M 400 90 L 400 190" stroke="#475569" stroke-width="2.5"
        marker-end="url(#arrow3)" class="diagram-path"/>
  <text x="410" y="145" font-size="13" font-weight="bold" fill="#475569"></text>

  <!-- : R/G_adm → S (bottom) -->
```

```
<?xml version="1.0" encoding="UTF-8"?>
<svg width="650" height="260" xmlns="http://www.w3.org/2000/svg">
  <defs>
    <style>
      @keyframes morphism-sweep {
        0% { opacity: 0; transform: translateX(-10px); }
        50% { opacity: 1; transform: translateX(0); }
        100% { opacity: 0; transform: translateX(10px); }
      }
      @keyframes divergence-pulse {
        0%, 100% { r: 3; fill: #059669; }
        50% { r: 5; fill: #34d399; }
      }
      .morphism { animation: morphism-sweep 3s ease-in-out infinite; }
      .stable-point { animation: divergence-pulse 2s ease-in-out infinite; }
    </style>
  </defs>

  <!-- Title -->
  <text x="180" y="25" font-size="16" font-weight="bold" fill="#1e293b">
    Quotient Stability Testing
  </text>

  <!-- Central realization -->
  <circle cx="100" cy="120" r="12" fill="#2563eb" stroke="#1e3a8a" stroke-width="2"/>
  <text x="95" y="125" font-size="11" font-weight="bold" fill="white">r</text>

  <!-- Morphism arrows g1-g8 -->
  <g class="morphism" style="animation-delay: 0s">
    <path d="M 112 120 L 170 120" stroke="#64748b" stroke-width="2"
          marker-end="url(#arrowgray)"/>
    <text x="135" y="115" font-size="10" fill="#64748b">g</text>
  </g>

  <g class="morphism" style="animation-delay: 0.3s">
    <path d="M 108 110 L 150 70" stroke="#64748b" stroke-width="2"
          marker-end="url(#arrowgray)"/>
    <text x="125" y="85" font-size="10" fill="#64748b">g</text>
  </g>

  <g class="morphism" style="animation-delay: 0.6s">
    <path d="M 100 108 L 100 50" stroke="#64748b" stroke-width="2"
          marker-end="url(#arrowgray)"/>
    <text x="105" y="75" font-size="10" fill="#64748b">g</text>
  </g>

  <g class="morphism" style="animation-delay: 0.9s">
    <path d="M 92 110 L 50 70" stroke="#64748b" stroke-width="2"
          marker-end="url(#arrowgray)"/>
    <text x="65" y="85" font-size="10" fill="#64748b">g</text>
  </g>

  <g class="morphism" style="animation-delay: 1.2s">
```

```
<?xml version="1.0" encoding="UTF-8"?>
<svg width="600" height="300" xmlns="http://www.w3.org/2000/svg">
  <defs>
    <style>
      @keyframes forward-pulse {
        0%, 100% { stroke-dashoffset: 0; }
        50% { stroke-dashoffset: -20; }
      }
      @keyframes reverse-pulse {
        0%, 100% { stroke-dashoffset: 0; }
        50% { stroke-dashoffset: 20; }
      }
      @keyframes error-grow {
        0%, 100% { r: 0; opacity: 0; }
        50% { r: 25; opacity: 0.3; }
      }
      .forward-path { animation: forward-pulse 2s linear infinite; }
      .reverse-path { animation: reverse-pulse 2s linear infinite; }
      .error-indicator { animation: error-grow 3s ease-in-out infinite; }
    </style>
  </defs>

  <!-- Title -->
  <text x="170" y="25" font-size="16" font-weight="bold" fill="#1e293b">
    Witness Integrity: Roundtrip Error
  </text>

  <!-- Initial realization r -->
  <circle cx="100" cy="100" r="10" fill="#2563eb" stroke="#1e3a8a" stroke-width="2"/>
  <text x="85" y="130" font-size="13" font-weight="bold" fill="#2563eb">r</text>
  <text x="70" y="145" font-size="10" fill="#64748b">Initial</text>

  <!-- Forward witness w -->
  <path d="M 120 100 Q 200 60 280 100" stroke="#059669" stroke-width="3"
        fill="none" marker-end="url(#arrowgreen)" stroke-dasharray="8,4"
        class="forward-path"/>
  <text x="190" y="70" font-size="12" font-weight="bold" fill="#059669">w</text>
  <text x="170" y="85" font-size="10" fill="#059669">forward</text>

  <!-- Transformed r' -->
  <circle cx="300" cy="100" r="10" fill="#059669" stroke="#065f46" stroke-width="2"/>
  <text x="282" y="130" font-size="13" font-weight="bold" fill="#059669">r'</text>
  <text x="265" y="145" font-size="10" fill="#64748b">Transformed</text>

  <!-- Reverse witness w¹ -->
  <path d="M 280 110 Q 200 150 120 110" stroke="#dc2626" stroke-width="3"
        fill="none" marker-end="url(#arrowred)" stroke-dasharray="8,4"
        class="reverse-path"/>
  <text x="185" y="165" font-size="12" font-weight="bold" fill="#dc2626">w¹</text>
  <text x="170" y="180" font-size="10" fill="#dc2626">inverse</text>

  <!-- Recovered r'' -->
  <circle cx="100" cy="120" r="10" fill="#dc2626" stroke="#991b1b" stroke-width="2"/>
```

```
<?xml version="1.0" encoding="UTF-8"?>
<svg width="650" height="320" xmlns="http://www.w3.org/2000/svg">
  <defs>
    <style>
      @keyframes f0-converge {
        0% { cy: 60; }
        100% { cy: 80; }
      }
      @keyframes f1-cycle {
        0%, 100% { cy: 160; }
        25% { cy: 140; }
        50% { cy: 160; }
        75% { cy: 180; }
      }
      @keyframes f2-drift {
        0% { cy: 240; }
        100% { cy: 280; }
      }
      @keyframes f3-chaos {
        0% { cy: 340; }
        10% { cy: 350; }
        20% { cy: 335; }
        30% { cy: 365; }
        40% { cy: 345; }
        50% { cy: 370; }
        60% { cy: 340; }
        70% { cy: 360; }
        80% { cy: 330; }
        90% { cy: 355; }
        100% { cy: 345; }
      }
      .f0-point { animation: f0-converge 3s ease-out infinite; }
      .f1-point { animation: f1-cycle 4s ease-in-out infinite; }
      .f2-point { animation: f2-drift 5s linear infinite; }
      .f3-point { animation: f3-chaos 3s linear infinite; }
    </style>
  </defs>

  <!-- Title -->
  <text x="200" y="20" font-size="16" font-weight="bold" fill="#1e293b">
    Flow Classification (F0-F3)
  </text>

  <!-- F0: Fixed Point -->
  <rect x="10" y="35" width="630" height="60" fill="#f0fdf4" stroke="#059669" rx="5"/>
  <text x="20" y="55" font-size="13" font-weight="bold" fill="#059669">
    F0: Fixed Point (Universal Invariant)
  </text>
  <line x1="50" y1="80" x2="610" y2="80" stroke="#d1d5db" stroke-width="1"/>
  <circle cx="50" r="4" fill="#3b82f6" class="f0-point"/>
  <circle cx="100" r="4" fill="#3b82f6" class="f0-point" style="animation-delay: 0.2s"/>
  <circle cx="150" r="4" fill="#3b82f6" class="f0-point" style="animation-delay: 0.4s"/>
  <circle cx="200" r="4" fill="#3b82f6" class="f0-point" style="animation-delay: 0.6s"/>
  <circle cx="250" r="4" fill="#059669" class="f0-point" style="animation-delay: 0.8s"/>
```