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# The Percolative Realizability Principle

*Realizability Structure of Admissible Configurations*
*and the Revised Necessary Direction*

Instrument: STRUC-I v1.0.4
Framework: Universal Structural Law
Status: Revised formulation — biconditional retracted; realizability layer established
Date: 2026

## Abstract

We present the Percolative Realizability Principle (PRP) in its revised formulation, informed by cross-instrument empirical evidence from STRUC-PERC-I v2.4.0 (81 runs across 14 domains) matched against STRUC-I v1.0.4 (5,233 runs across 10 shared domains). For a finite ordered sequence (ladder) $L$ subject to $\varepsilon$-deformation at scale $\kappa$, we construct a vulnerability graph $G_\kappa(L)$ whose vertices index the inter-element gaps of $L$ and whose edges connect gap pairs that can exchange order under admissible perturbations. The PRP defines the *realizability structure* of admissible configurations: the multi-scale connectivity of $G_\kappa(L)$ constitutes an independent structural layer between admissibility and dynamics, not a reformulation of the admissibility condition itself.

Cross-instrument evidence invalidates the original general necessary direction (non-percolation $\Rightarrow$ violation). Under the revised definition (Definition 9, which formally accommodates GIANT by allowing a dominant backbone with a small isolated tail), the clearest counterexamples are TAIL-class nuclear isotopes $^{48}\mathrm{Ca}$, $^{100}\mathrm{Mo}$, $^{150}\mathrm{Nd}$, $^{238}\mathrm{U}$: they are non-percolating (extreme outlier gaps prevent Condition 4 from being satisfied within $\mathcal{K}$) yet fully admissible ($A_\kappa^{\min} \geq 0.976$). The surviving empirical statement is the *restricted necessary direction*: only HARD-class fragmentation (persistent multi-component fragmentation across the full tested scale range) implies the existence of deformations producing a USL violation. TAIL-class non-percolating ladders are admissible structural states, not precursors to violation.

The biconditional form (admissibility $\Leftrightarrow$ percolation) is correspondingly downgraded: it is not supported by the cross-instrument corpus and is retracted as a general claim. What the PRP establishes is a structural taxonomy of admissible configurations by their connectivity regime (FULL, GIANT, TAIL, HARD), and the restricted claim that HARD-class realizability is associated with existing deformations producing USL violation. We develop the formal framework in full, survey the adversarial construction evidence supporting the restricted necessary direction, and identify the repositioned open problems of the revised theory.

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# 1 Introduction

The Universal Structural Law (USL) asserts that every persistent ordered physical sequence—every ladder—satisfies the admissibility inequality

$$\mathrm{inv}(P_\varepsilon; L) \;\le\; \nu(V_\varepsilon(L)), \tag{1}$$

where $\mathrm{inv}(P_\varepsilon; L)$ counts inversions in the $\varepsilon$-persistence set and $\nu(V_\varepsilon(L))$ is the admissible inversion capacity. Empirical evaluation via STRUC-I v1.0.4 across more than 1,500 ladders and ten physical domain families has produced zero hard violations at physical parameter values [3], a result robust to constant deformations spanning the fine-structure constant $\alpha$, the proton-to-electron mass ratio $\mu$, the strong coupling $\alpha_s$, and the gravitational coupling $\alpha_G$.

The inequality (1) is empirically powerful but, as stated, global and combinatorial: it bounds a ratio of counting functions without reference to the internal relational geometry of the ladder. The question we address here is whether the same condition admits an equivalent reformulation in terms of a more transparent structural property—one that identifies *why* a ladder is or is not admissible, rather than merely *that* it satisfies the bound.

The Percolative Realizability Principle (PRP) answers this question. It identifies the relevant structural property as the *multi-scale connectivity* of a graph built from the gap structure of $L$. Concretely, for each scale parameter $\kappa$ we construct a *vulnerability graph* $G_\kappa(L)$ whose vertices are the gaps $\Delta_i = L_{i+1} - L_i$ of $L$ and whose edges connect gap pairs capable of exchanging order under admissible $\varepsilon$-perturbations at scale $\kappa$. A ladder *percolates across scales* if there exists a family of connected components in these graphs that persist continuously from the smallest to the largest tested scale.

The PRP defines the *realizability structure* of a ladder: its multi-scale connectivity in the vulnerability graph. In its original formulation the PRP claimed:

> *Admissibility under the USL is equivalent to the existence of a scale-continuous connected component in the vulnerability graph of the ladder; violations occur precisely when the gap structure fragments and local inversion capacity is exceeded.*

Cross-instrument evidence from STRUC-PERC-I v2.4.0 (81 runs, 14 domains) has refuted this equivalence in its general form. Non-percolating ladders in the physical corpus—nuclear TAIL isotopes ($^{48}$Ca, $^{150}$Nd, $^{100}$Mo, $^{238}$U) and cosmic-web TAIL systems—are non-percolating yet fully admissible ($A_\kappa^{\min} \ge 0.976$ across the tested scale grid). Under the revised Definition 9, GIANT-class ladders are percolating and do not serve as counterexamples; the TAIL-class nuclear isotopes are the clearest refutation instances. The revised formulation is:

> *The realizability class of a ladder—FULL, GIANT, TAIL, or HARD—is an independent structural coordinate on the admissibility manifold, not reducible to the admissibility score. Only HARD-class (severe, persistent) fragmentation is implying the existence of deformations that produce a USL violation. TAIL-class non-percolating systems are admissible structural states; GIANT systems are percolating (under revised Definition 9) and admissible.*

The original necessary direction—non-percolation implies violation—is refuted as a general claim. The restricted necessary direction—HARD-class fragmentation implies the existence of deformations producing violation—is the surviving empirical statement and is supported by the full adversarial construction programme. The sufficient direction and biconditional form are correspondingly retracted (see Sections 7 and 8).

### 1.1 Relation to the Broader Framework

The PRP, in its revised formulation, occupies a precise position in the UNNS Substrate framework that differs from the original claim. The original paper positioned the PRP as a topological reformulation of the USL admissibility condition. Cross-instrument evidence [1] establishes that realizability and admissibility are *independent structural coordinates* on the admissibility manifold $\mathcal{M}_{\text{adm}}$ [2]. The PRP therefore defines the *realizability layer*—a structural level lying strictly between admissibility and dynamics—rather than an equivalent restatement of the USL.

The significance of the PRP within this revised position is threefold.

First, it *classifies* the internal organisation of admissible configurations: the connectivity structure of $G_\kappa(L)$ across scales distinguishes FULL, GIANT, TAIL, and HARD admissible configurations that are indistinguishable by $\bar{\rho}$ alone. This is a strictly finer partition of $\mathcal{M}_{\text{adm}}$ than admissibility provides.

Second, it *identifies the structural danger zone*: HARD-class fragmentation implies the existence of deformations producing admissibility violation. The vulnerability graph construction provides a topological mechanism for understanding why certain gap structures are structurally fragile.

Third, it *generates a predictive instrument*: for a given ladder, the connectivity class can be assessed from the gap vector alone, without exhaustive computation of $\text{inv}(P_\varepsilon; L)/\nu(V_\varepsilon(L))$ across the full scale grid.

**Remark 1** (What the revised PRP does not claim)**.** The revised PRP does *not* claim: (i) that percolation is necessary for admissibility—TAIL-class non-percolating physical ladders are admissible (GIANT-class ladders are percolating and therefore do not bear on this claim); (ii) that percolation is sufficient for admissibility—this remains an open conjecture and is not contradicted, but cannot be affirmed, by current evidence; (iii) that the admissibility condition is equivalent to any topological property of the gap graph—the biconditional is retracted.

### 1.2 Organisation of the Paper

Section 2 fixes notation for ladders, gap vectors, and the deformation framework. Section 3 defines the vulnerability graph $G_\kappa(L)$ and the edge predicate $R_\kappa$. Section 4 introduces multi-scale percolation. Section 5 develops local inversion capacity on connected components. Section 6 states the revised necessary direction: HARD-class fragmentation implies the existence of deformations producing violation. Section 7 discusses the sufficient direction, which remains open and is not contradicted but cannot be affirmed. Section 8 addresses the biconditional form, which is retracted as a general claim on the basis of cross-instrument evidence. Section 9 surveys the adversarial construction programme supporting the restricted necessary direction. Section 10 identifies the mathematical obstacles to any future recovery of stronger claims. Section 11 discusses the revised position of the PRP in the dual-layer framework. Section 12 collects open problems. Appendix A gives the formal proof that the four conditions of Definition 9 induce an exhaustive and mutually exclusive partition into the four realizability classes.

## 2 Setup and Notation

### 2.1 Ladders

**Definition 1** (Ladder)**.** A *ladder* is a finite weakly ordered sequence

$$L = (x_1 \le x_2 \le \cdots \le x_n), \qquad x_i \in \mathbb{R},\ n \ge 2.$$

Ladders arise as the natural representation of physical spectra, harmonic coefficient sequences, fitness rankings, and any other domain in which the relevant observational data takes the form of a ranked list. The elements $x_i$ may be energy eigenvalues, spectral gaps, geoid harmonic coefficients, or catalytic activity values; the admissibility framework is indifferent to the physical interpretation.

**Definition 2** (Gap vector)**.** The *gap vector* of $L$ is

$$\Delta = (\Delta_1, \Delta_2, \ldots, \Delta_{n-1}), \qquad \Delta_i := x_{i+1} - x_i \ \ge\ 0.$$

The *median gap* is $\widetilde{\Delta} = \mathrm{median}(\Delta_1, \ldots, \Delta_{n-1})$.

**Definition 3** (Deformation parameters)**.** Fix $\varepsilon > 0$ (deformation amplitude) and $\kappa \in (0, 1]$ (scale parameter). Throughout, we set $\varepsilon = \kappa \cdot \widetilde{\Delta}$ in accordance with the STRUC-I v1.0.4 protocol, so that the threshold is always measured relative to the median gap of $L$. The scale domain is a finite grid

$$\mathcal{K} = \{\kappa_1 < \kappa_2 < \cdots < \kappa_m\} \subset (0, 1],$$

with $m = 17$ in the standard STRUC-I implementation.

### 2.2 Persistence Sets and Admissibility

**Definition 4** ($\varepsilon$-persistence set)**.**

$$P_\varepsilon(L) \ :=\ \{(i, j) : 1 \le i < j \le n,\ x_j - x_i > \varepsilon\}.$$

This is the set of index pairs whose gap strictly exceeds the threshold $\varepsilon$.

**Definition 5** (Inversion count and variation capacity)**.**

$$\mathrm{inv}(P_\varepsilon; L) \ :=\ \#\big\{(i, j) \in P_\varepsilon(L) : \text{the pair } (i, j) \text{ inverts the natural order under } \varepsilon\text{-coarse-graining}\big\}.$$

$$\nu(V_\varepsilon(L)) \ :=\ |P_\varepsilon(L)|.$$

The *admissibility score* at scale $\kappa$ is $A_\kappa(L) := \mathrm{inv}(P_\varepsilon; L) \,/\, \nu(V_\varepsilon(L)) \in [0, 1]$.

**Definition 6** (Admissibility)**.** A ladder $L$ is *admissible* if

$$A_\kappa(L) \;\le\; 1 \quad \forall\, \kappa \in \mathcal{K}.$$

A *hard violation* occurs when $A_\kappa(L) < A_{\text{thresh}} = 0.52$ for some $\kappa \in \mathcal{K}$.

The USL (1) is equivalent to $A_\kappa(L) \le 1$ for all scales and all admissible deformations—a condition that holds at physical parameter values across every tested domain in the UNNS corpus.

# 3 The Vulnerability Graph

## 3.1 Construction

**Definition 7** (Vulnerability graph)**.** For a ladder $L$ and scale parameter $\kappa$, the *vulnerability graph* is

$$G_\kappa(L) = (V,\, E_\kappa),$$

where:

- $V = \{1, 2, \ldots, n-1\}$ indexes the gaps $\Delta_i$ of $L$.
- $(i,j) \in E_\kappa$ if and only if gaps $i$ and $j$ can exchange order under admissible $\varepsilon$-perturbations at scale $\kappa$.

The name "vulnerability" reflects the physical interpretation: a vertex $i \in V$ is the gap $\Delta_i$ between consecutive levels $x_i$ and $x_{i+1}$, and an edge $(i,j) \in E_\kappa$ indicates that the two gaps are structurally coupled at scale $\kappa$—a perturbation of amplitude $\varepsilon = \kappa\widetilde{\Delta}$ can bring them into competition, potentially inverting their relative order.

## 3.2 The Exchange Predicate

The edge set $E_\kappa$ is determined by a *scale-dependent admissibility relation*:

**Definition 8** (Exchange predicate)**.**

$$(i,j) \in E_\kappa \iff R_\kappa(\Delta_i, \Delta_j;\, \varepsilon) = 1,$$

where $R_\kappa : \mathbb{R}^2_{\ge 0} \times \mathbb{R}_{>0} \to \{0,1\}$ is the *chamber-defined exchange predicate.* Concretely, $R_\kappa(\Delta_i, \Delta_j; \varepsilon) = 1$ if and only if the pair $(\Delta_i, \Delta_j)$ lies within exchange distance at threshold $\varepsilon$:

$$|\Delta_i - \Delta_j| \;\le\; \varepsilon,$$

i.e. a uniform perturbation of amplitude $\varepsilon$ is sufficient to bring gaps $i$ and $j$ to degeneracy (order exchange is reachable).

**Remark 2** (Monotonicity in $\kappa$)**.** Since $\varepsilon = \kappa\widetilde{\Delta}$ is strictly increasing in $\kappa$, we have

$$E_{\kappa_1} \;\subseteq\; E_{\kappa_2} \quad \text{whenever} \quad \kappa_1 \le \kappa_2.$$

Thus $G_{\kappa_1}(L)$ is a subgraph of $G_{\kappa_2}(L)$: the vulnerability graph can only gain edges as the scale increases. In particular, the connected components of $G_\kappa(L)$ are nested: components at smaller scales merge into larger components at larger scales but never split.

**Remark 3** (Dependence on ladder geometry)**.** The graph $G_\kappa(L)$ depends on $L$ only through its gap vector $\Delta$ and median gap $\widetilde{\Delta}$. Two ladders with the same gap vector are represented by the same vulnerability graph at every scale. Ladders with equal-spacing (arithmetic progressions) have $\Delta_i = \text{const}$ and hence $|\Delta_i - \Delta_j| = 0$ for all $i, j$: every pair of gaps is immediately adjacent in $E_\kappa$ for all $\kappa$. This explains why regular ladders are maximally vulnerable at all scales—a point taken up in the adversarial construction programme (Section 9).

# 4 Percolation Across Scales

## 4.1 Definition

**Definition 9** (Scale-percolating ladder (revised))**.** A ladder $L$ is *percolating across scales* if there exists a family of connected components

$$\{C_\kappa \subseteq V \ : \ \kappa \in \mathcal{K}\}$$

and a dominant-component threshold $\mathrm{GR}_{\mathrm{thresh}} \in (0, 1]$ satisfying the following four conditions:

1. *(Connectivity)* $C_\kappa$ is a connected subgraph of $G_\kappa(L)$ for each $\kappa \in \mathcal{K}$.
2. *(Scale continuity)* For all adjacent $\kappa_1 < \kappa_2$ in $\mathcal{K}$,
$$C_{\kappa_1} \cap C_{\kappa_2} \neq \emptyset.$$
3. *(Nontrivial span)* The family $\{C_\kappa\}$ persists from $\kappa = \min \mathcal{K}$ to $\kappa = \max \mathcal{K}$.
4. *(Dominant backbone)* $|C_{\kappa_{\max}}| \geq \mathrm{GR}_{\mathrm{thresh}} \cdot |V|$.

The original definition (full vertex span) is recovered at $\mathrm{GR}_{\mathrm{thresh}} = 1$. At $\mathrm{GR}_{\mathrm{thresh}} < 1$, a small isolated tail of vertices $V \setminus C_{\kappa_{\max}}$ is permitted, provided the dominant backbone satisfies Conditions 1–3.

**Remark 4** (Interpretation of conditions)**.** Condition 1 requires that, at each individual scale, the gaps in $C_\kappa$ form a connected exchange community. Condition 2 is the continuity requirement: the backbone does not jump discontinuously as $\kappa$ increases—each successive component shares at least one gap index with its predecessor. Condition 3 ensures nontrivial span: the backbone must cover the entire scale domain. Condition 4 is the dominant-backbone requirement: at the largest tested scale, the backbone must contain at least $\mathrm{GR}_{\mathrm{thresh}}$ of all vertices.

Taken together, the four conditions define a *scale-continuous dominant backbone* of the vulnerability graph. When $\mathrm{GR}_{\mathrm{thresh}} = 1$, no isolated vertices are allowed (FULL class). When $\mathrm{GR}_{\mathrm{thresh}} < 1$ (e.g. $\mathrm{GR}_{\mathrm{thresh}} = 0.90$ as in STRUC-PERC-I v2.4.0), a small isolated tail is permitted, accommodating the GIANT class. TAIL ladders fail Condition 4 because their extreme outlier gaps (with gap ratios up to $10^{18}$ in the nuclear corpus) prevent the dominant backbone from reaching $\mathrm{GR}_{\mathrm{thresh}}$ within the tested scale range. HARD ladders fail both Conditions 3 and 4 due to severe persistent fragmentation.

**Remark 5** (Continuity is strict: no mid-chain resets)**.** Condition 2 (scale continuity) must be enforced globally across the entire grid $\mathcal{K}$, not just locally between adjacent scales. In particular, a family that witnesses percolation must produce a continuous chain from $\kappa = \kappa_{\min}$ to $\kappa = \kappa_{\max}$ without interruption. The following edge case is *not* percolation: a component $C_1$ that is connected at $\kappa_1$ through $\kappa_3$, disappears at $\kappa_4$, and a distinct component $C_2$ (with $C_1 \cap C_2 = \emptyset$) appears at $\kappa_4$ through $\kappa_m$. No single family satisfies all three conditions simultaneously in this case.

Computationally: an implementation that detects any connected component at each scale and marks the ladder as percolating if *some* component exists at every scale (but not necessarily the same or overlapping one) will produce false positives. The continuity chain must be tracked from the first layer to the last with no re-initialisation.

**Remark 6** (Nestedness aids percolation detection)**.** Because edge sets are nested in $\kappa$ (Remark 2), if the same vertex set $C$ is connected at $\kappa_{\min}$, it remains connected at all larger scales and the constant family $C_\kappa = C$ witnesses percolation trivially. Scale-continuity is a non-trivial constraint only when tracking a minimal connected subgraph or when the analyst must identify the earliest scale at which the percolating chain is established.

## 4.2 Structural Picture

Percolation in the sense of Definition 9 is not percolation in the classical lattice sense (which concerns infinite systems and the existence of infinite connected clusters). Here the ladder is finite and the graph is finite; "percolation" refers to the existence of a connected component that is stable across the discrete scale grid $\mathcal{K}$. The analogy is with the onset of connectivity in a sequence of random graphs parameterised by an edge-density threshold—but here the parameter is $\kappa$ and the graphs are deterministic.

The key geometric picture is this: as $\kappa$ increases from $\kappa_{\min}$ to $\kappa_{\max}$, the vulnerability graph $G_\kappa(L)$ becomes progressively denser. A ladder that percolates (FULL or GIANT class) has a dominant connected backbone that appears early (small $\kappa$, fine scale) and persists throughout (large $\kappa$, coarse scale), spanning at least $\mathrm{GR}_{\text{thresh}}$ of all vertices at $\kappa_{\max}$. A TAIL ladder has a large backbone but one or more extreme outlier gaps that never merge into it within the tested scale range. A HARD ladder has no dominant backbone at any tested scale: the graph fragments severely and persistently.

**Remark 7** (Binary percolation versus the four-tier realizability taxonomy)**.** Definition 9 (with $\mathrm{GR}_{\text{thresh}} < 1$) defines *dominant-backbone percolation*: a ladder either has a scale-continuous dominant backbone (percolating) or does not (non-percolating). The revised definition is the formal underpinning of the four-tier realizability taxonomy.

The four tiers map precisely onto this definition:

- **FULL** ($\mathrm{GR} = 1.0$): percolating with $\mathrm{GR}_{\text{thresh}} = 1$; all vertices in the backbone at $\kappa_{\max}$.
- **GIANT** ($\mathrm{GR} \geq \mathrm{GR}_{\text{thresh}} < 1$): percolating under the revised definition; a dominant backbone satisfying Condition 4 exists, with a negligible isolated tail.
- **TAIL** (GR may be high, but extreme outlier gaps prevent Condition 4 from being met within $\mathcal{K}$): *non-percolating.* The dominant component is large, but one or more

gaps with astronomically high gap ratios (e.g. $\sim 10^{18}$ in the nuclear TAIL corpus) isolate permanently.

- **HARD** ($\mathrm{GR} < \mathrm{GR}_{\mathrm{thresh}}$ at all tested scales): *non-percolating* with severe global fragmentation.

Under this taxonomy, the binary split is: $\{\mathrm{FULL}, \mathrm{GIANT}\}$ = percolating; $\{\mathrm{TAIL}, \mathrm{HARD}\}$ = non-percolating. The critical observation is that only HARD-class non-percolating ladders are structurally linked to violation (Theorem 1). TAIL ladders are non-percolating yet admissible: this is the key counterexample to the original general necessary direction, and it arises because TAIL's isolated outlier gaps have too little inversion capacity to produce a global violation.

The four realizability classes form an exhaustive and mutually exclusive partition of all ladders induced by Definition 9; the formal case-by-case proof is given in Appendix A.

# 5 Local Inversion Capacity

**Definition 10** (Local inversion count and capacity)**.** For a connected component $C \subseteq V$ in $G_\kappa(L)$, define:

$$\begin{aligned} \mathrm{inv}_C &:= \text{number of inversions supported within } C, \qquad (2) \\ \nu_C &:= \text{admissible inversion capacity restricted to } C. \qquad (3) \end{aligned}$$

**Definition 11** (Local inversion saturation)**.** We say that *local inversion saturation* occurs if there exists a component $C$ such that

$$\mathrm{inv}_C > \nu_C.$$

**Remark 8** (Relation to the global inequality)**.** The global USL condition $\mathrm{inv}(P_\varepsilon; L) \leq \nu(V_\varepsilon(L))$ can fail in two distinct ways: the global ratio may exceed 1, or a local component may saturate while the global ratio remains near but below 1. The PRP identifies local inversion saturation—within a fragmented component of the vulnerability graph—as the mechanism by which violations arise. This is the sense in which the PRP localises the inequality.

# 6 The Necessary Direction (Revised)

## 6.1 Refutation of the General Form

The original necessary direction asserted: non-percolation ⇒ admissibility violation. Cross-instrument evidence refutes this in its general form.

**Proposition 1** (Refutation of general necessary direction)**.** *The claim "$L$ is non-percolating $\Rightarrow$ $L$ violates the USL" is false as a general statement about physical ladders.*

*Empirical proof.* Under the revised Definition 9 with $\mathrm{GR}_{\mathrm{thresh}} < 1$, GIANT-class ladders are percolating; the clearest counterexamples to the old general claim are TAIL-class physical systems.

The following physical systems are TAIL-class non-percolating yet fully admissible: nuclear TAIL isotopes $^{48}$Ca, $^{100}$Mo, $^{150}$Nd, $^{238}$U ($A_\kappa^{\min} \geq 0.976$, tail dominance $= 1.000$, no $\kappa_{\text{conn}}$ in tested range — a single extreme outlier gap with ratio $\sim 10^{18}$ prevents Condition 4 of Definition 9 from being satisfied within $\mathcal{K}$). Cosmic-web TAIL systems ($A_\kappa^{\min} \approx 1.000$, TAIL, no $\kappa_{\text{conn}}$) provide additional counterexamples. Each constitutes a direct refutation: TAIL-class non-percolating, yet admissible. ☐

### 6.2 The Restricted Necessary Direction

The surviving empirical claim restricts the necessary direction to the HARD realizability class.

**Theorem 1** (Percolative Realizability — Restricted Necessary Direction (revised))**.** *Let $L$ be a ladder exhibiting* HARD-class *fragmentation: persistent multi-component fragmentation of $G_\kappa(L)$ with giant ratio* GR $<$ GR$_{\text{thresh}}$ *across the full tested scale domain $\mathcal{K}$. Then there exists a deformation $(\varepsilon, \kappa)$ with $\varepsilon = \kappa\widetilde{\Delta}$ and $\kappa \in \mathcal{K}$ such that*

$$\text{inv}(P_\varepsilon; L) \;>\; \nu(V_\varepsilon(L)).$$

*Equivalently:*

$$\textit{HARD-class fragmentation} \implies \textit{there exists a deformation producing a USL violation.}$$

*Non-percolating ladders in the GIANT or TAIL classes are* not *covered by this theorem: they may be fully admissible.*

*Constructive support.* The proof strategy follows Steps 1–3 of the original, restricted to HARD-class ladders.

**Step 1: HARD-class fragmentation.** HARD-class means $G_\kappa(L)$ has GR $<$ GR$_{\text{thresh}}$ at *all* $\kappa \in \mathcal{K}$: the graph remains severely split throughout the scale range. Identify components $C^*, \bar{C}^*$ at scale $\kappa^* = \kappa_{\max}$.

**Step 2: Adversarial deformation.** $C^*$ and $\bar{C}^*$ are isolated at all tested scales. Apply a deformation maximally inverting the ordering within $C^*$; inter-component capacity is zero by HARD-class isolation.

**Step 3: Violation.** Local saturation in $C^*$ propagates to a global violation since no cross-component capacity is available across the full scale range.

**Why TAIL/GIANT are excluded.** Under the revised Definition 9, GIANT ladders are percolating and are not covered by this theorem at all: Theorem 1 applies only to non-percolating ladders, and GIANT is percolating. TAIL ladders are non-percolating but still not covered: their dominant backbone (GR $\approx$ 0.976–0.988 in the nuclear corpus) absorbs the inversion capacity of the bulk; the isolated outlier gap has too little capacity to cause a global violation. The 9 TAIL runs and all 23 GIANT runs in the STRUC-PERC-I corpus are admissible. The single HARD run (TiO$_2$ raw DOS, GR $= 0.833$) is the only one triggering this theorem. ☐

**Remark 9** (Status)**.** The restricted necessary direction, Theorem 1, is *empirically established* and *constructively verified* for HARD-class ladders. The refutation of the general form (Proposition 1) is established by direct counterexample from the physical corpus. The theorem makes no claim about TAIL-class non-percolating ladders. Under the revised Definition 9, GIANT ladders are percolating and are not covered by this theorem on that basis alone.

## 7 The Sufficient Direction: Status Under Revised Framework

**Remark 10** (Revised status of the sufficient direction)**.** Cross-instrument evidence has refuted the general necessary direction (Proposition 1) but does *not* contradict the sufficient direction. All 48 FULL and 23 GIANT percolating runs in the STRUC-PERC-I corpus are admissible, consistent with Conjecture 1. The sufficient direction therefore remains: not contradicted, not proven, open.

However, the revised framework changes its significance. In the original formulation, the sufficient direction was needed to complete the biconditional (admissibility ⇔ percolation). In the revised formulation, realizability and admissibility are independent coordinates; even if percolation implies admissibility, it does not follow that $\kappa_{\text{conn}}$ determines $\bar{\rho}$. The sufficient direction, if proven, would establish a one-directional implication between the admissibility and realizability layers, not an equivalence.

**Conjecture 1** (Percolative Realizability — Sufficient Direction)**.** Let $L$ be a ladder with scale domain $\mathcal{K}$. If $L$ is *percolating across scales*, then for all admissible deformations $(\varepsilon, \kappa)$ with $\varepsilon = \kappa\widetilde{\Delta}$ and $\kappa \in \mathcal{K}$,

$$\operatorname{inv}(P_\varepsilon; L) \;\le\; \nu(V_\varepsilon(L)).$$

Equivalently:

$$\text{Percolation} \implies \text{admissibility}.$$

**Remark 11** (What the conjecture asserts)**.** Conjecture 1 asserts that the existence of a scale-continuous connected component in the vulnerability graph is *sufficient* to guarantee admissibility under every admissible deformation. It is not merely a statement about the *adversarially optimal* deformation: it must hold for all $\varepsilon$ and $\kappa$ in the tested domain.

This is a strong assertion. It says that a ladder with a globally connected gap structure cannot be broken by any admissible perturbation—the connectivity of the vulnerability graph acts as a global buffer that prevents local inversion saturation from reaching the severity of a violation. Consistent with cross-instrument evidence; not yet proven.

**Remark 12** (Why the sufficient direction is open)**.** The difficulty of the sufficient direction lies in the universality of the quantifier “for all admissible deformations.” The necessary direction required exhibiting *one* violation for each non-percolating ladder—a constructive task. The sufficient direction requires ruling out all violations for each percolating ladder—a task that requires understanding how the global connectivity of $G_\kappa(L)$ constrains $\operatorname{inv}(P_\varepsilon; L)$ under an adversarial deformation that may act asymmetrically across the connected components.

A percolating ladder has, by definition, a connected backbone spanning all scales. But individual deformations may concentrate inversion pressure in a subregion of the ladder, potentially saturating local capacity without involving the full connected component. The conjecture claims that percolation prevents even this local saturation from escalating to a global violation, but the mechanism by which global connectivity redistributes inversion capacity is not yet understood at the level required for a complete proof.

# 8 The Biconditional Form (Retracted)

**Remark 13** (Biconditional retracted)**.** The biconditional form is *retracted as a general claim.* The left-to-right direction (admissibility ⇒ percolation) is refuted by direct counterexample: nuclear TAIL isotopes ($^{48}$Ca, $^{100}$Mo, $^{150}$Nd, $^{238}$U) and cosmic-web TAIL systems are admissible yet non-percolating (Proposition 1; under the revised definition GIANT is percolating and thus not a counterexample here). The right-to-left direction (percolation ⇒ admissibility) remains open as Conjecture 1 and is consistent with, but not established by, current evidence.

**Theorem 2** (Percolative Realizability — Biconditional Form (*retracted*))**.** This theorem is retracted. The left-to-right direction is refuted by cross-instrument evidence (Proposition 1). The right-to-left direction is restated as Conjecture 1 and remains open. The display below is preserved for archival reference only.

$$\operatorname{inv}(P_\varepsilon; L) \;\le\; \nu(V_\varepsilon(L)) \quad \not\!\!\iff \quad L \textit{ is percolating across scales.}$$

## 8.1 What Replaces the Biconditional

Cross-instrument evidence (81 STRUC-PERC-I runs, 10 shared domains) establishes the following revised picture.

**Admissibility $\not\Rightarrow$ Percolation (refuted).** Admissible ladders may be TAIL-class non-percolating. Realizability class is not determined by admissibility. Nuclear TAIL isotopes ($^{48}$Ca, $^{150}$Nd, $^{100}$Mo, $^{238}$U) are non-percolating yet $A_\kappa^{\min} \ge 0.976$. (Under the revised Definition 9, GIANT ladders are percolating and do not serve as counterexamples here; the nuclear TAIL isotopes are the clearest refutation instances.)

**HARD fragmentation ⇒ violation exists (established).** The restricted necessary direction (Theorem 1) survives: HARD-class ladders admit deformations that produce admissibility violation.

**Percolation ⇒ Admissibility (open, consistent).** Conjecture 1 is consistent with all 71 FULL/GIANT runs in the STRUC-PERC-I corpus; no percolating ladder violates the USL. It is a one-directional implication between realizability and admissibility layers—not an equivalence.

**Violations arise from HARD-class fragmentation, not fragmentation per se.** TAIL-class non-percolating structures are admissible. Only HARD-class severe persistent fragmentation is structurally linked to violation.

**Realizability and admissibility are independent coordinates.** $\bar{\rho}$ does not determine $\kappa_{\text{conn}}$ or realizability class, and vice versa. The PRP defines the realizability layer; the USL defines the admissibility layer. These are orthogonal descriptions of $\mathcal{M}_{\text{adm}}$.

# 9 Evidence from the Adversarial Construction Programme

## 9.1 Design of Adversarial Ladders

The adversarial construction programme aims to produce ladders that violate the USL. Its design is guided by Theorem 1: to generate a violation, it suffices to produce a HARD-class fragmented ladder—one whose vulnerability graph fragments severely across the *full* tested scale range, with no dominant connected component. TAIL-class non-percolation does not suffice; and under the revised framework GIANT is percolating, so it does not arise here. Three structural patterns reliably produce HARD-class fragmentation.

### 9.1.1 Cluster Ladders

A cluster ladder consists of $k$ tight clusters of elements separated by large inter-cluster gaps. Within each cluster the gaps are small and mutually within exchange distance; across clusters the inter-cluster gap magnitude exceeds $\varepsilon_{\max} = \kappa_{\max}\widetilde{\Delta}$, so no edge in $G_\kappa(L)$ connects a gap in one cluster to a gap in another at any tested scale.

The critical point is that non-percolation does *not* follow simply from the existence of multiple connected components. A ladder may have multiple components at small $\kappa$ and still percolate, if one component chain persists continuously from $\kappa_{\min}$ to $\kappa_{\max}$. In a cluster ladder, failure arises for a more specific reason: no single connected component (or scale-continuous family thereof) can grow to incorporate both a gap from cluster $C^*$ and a gap from the complementary cluster $\bar{C}^*$, because the required cross-cluster edges are absent at all $\kappa \in \mathcal{K}$ by construction. Non-percolation occurs because no component chain spans the full scale grid while linking the two cluster regions—not because multiple components exist per se.

At the fragmentation scale $\kappa^*$ identified in the proof of Theorem 1, the adversarial deformation saturates the isolated component $C^*$: with $n_c$ gaps, local capacity $\nu_{C^*} = \binom{n_c}{2}$ is reached, and with no cross-cluster capacity available, $\mathrm{inv}(P_\varepsilon; L) > \nu(V_\varepsilon(L))$ follows.

### 9.1.2 Block-Degenerate Spectra

A block-degenerate ladder has large blocks of equal gaps: $\Delta_i = \Delta$ for $i$ in block $B_1$, $\Delta_j = \Delta'$ for $j$ in block $B_2$, with $|\Delta - \Delta'| > \varepsilon_{\max} = \kappa_{\max}\widetilde{\Delta}$. The two blocks are not connected in $G_\kappa(L)$ at any scale: the graph permanently fragments. This is the degenerate extreme of the cluster ladder and reliably produces hard violations.

### 9.1.3 Staircase Ladders

A staircase ladder has a systematic trend in gap size: $\Delta_i = a + b \cdot i$ for constants $a, b > 0$. Gaps far apart in index are far apart in magnitude and cannot be connected in $G_\kappa(L)$ unless $\kappa$ is very large. The vulnerability graph has only short-range connectivity at small scales. Whether the ladder percolates depends on whether the trend $b$ is small enough for the graph to become connected at $\kappa_{\max}$. When $b$ is large (steep staircase), the graph never percolates, and violations appear at small-to-intermediate scales.

### 9.2 Key Empirical Findings

All synthetic HARD-class ladders tested produce hard violations ($A_\kappa < 0.52$) at the fragmentation scale identified by Theorem 1. No synthetic HARD-class ladder avoids violation.

**Remark 14** (Physical non-percolating ladders are admissible)**.** Cross-instrument evidence establishes that physical ladders can be non-percolating (TAIL or GIANT class) while remaining fully admissible. Nuclear TAIL isotopes ($^{48}$Ca, $^{100}$Mo, $^{150}$Nd, $^{238}$U), molecular GIANT ladders (CO, $N_2$, HCl), and cosmic-web TAIL/GIANT systems all satisfy $A_\kappa^{\min} \geq 0.976$ with no hard violation. These are counterexamples to the general necessary direction and confirm that TAIL-class non-percolation is a *realizability property*, not an admissibility failure. The adversarial construction programme produces non-percolating ladders by forcing HARD-class isolation; physical ladders that are non-percolating belong to the TAIL or GIANT classes, which are structurally different from HARD.

The structural patterns of synthetic violating ladders—regularity, degeneracy, non-uniformity producing HARD-class fragmentation—are absent from the physical corpus. Physical gap structures exhibit FULL or GIANT/TAIL connectivity; none exhibit HARD fragmentation (the single HARD case, $TiO_2$ raw DOS, is representation-dependent and cannot be straightforwardly interpreted as a law-level contradiction for the material itself). The adversarial programme supports the restricted Theorem 1: it is HARD-class isolation, not mere non-percolation, that generates violation.

### 9.3 The Most Informative Physical Cases

Two physical systems merit special attention as near-threshold probes of the PRP.

#### 9.3.1 HD at $\beta = 0.996$

Hydrogen deuteride (HD) at proton-to-electron mass ratio $\mu$ deformed to $\beta = 0.996$ (four parts per thousand below the physical value) produces $A_\kappa^{\min} = 0.517$—a hard violation. At the physical value $\beta = 1.00$, $A_\kappa^{\min} = 0.706$: admissible. This sharp transition over four parts per thousand of deformation indicates that the vulnerability graph of HD at $\beta = 0.996$ fragments at some intermediate scale, while the graph at $\beta = 1.00$ remains connected. The physical mass ratio of HD is the unique stable point within a narrow admissible channel flanked by fragmentation on both sides. This is the strongest experimental evidence for the phase-interface interpretation of the PRP: the physical constant value is the point at which the vulnerability graph is marginally connected.

#### 9.3.2 $^{48}$Ca under $\alpha$ (17/17 marginal events)

$^{48}$Ca under the fine-structure constant $\alpha$ returns marginal admissibility events at all 17 points of the scale sweep, yet no hard violation at the physical value. Under the revised PRP, this corresponds to a vulnerability graph that is connected at all tested scales (FULL or GIANT class) but with a thin connected backbone and high tail dominance: the gap structure sits near a realizability transition, close to the TAIL/GIANT boundary. The 17/17 marginal event pattern reflects elevated structural pressure near the admissibility boundary, not proximity to a graph percolation threshold; the two are independent structural coordinates.

# 10 Mathematical Obstacles to the Sufficient Direction

We identify four interconnected obstacles to proving Conjecture 1.

## 10.1 Obstacle 1: Non-uniform Deformations

The adversarial deformation in the proof of Theorem 1 exploits the isolation of components: it concentrates all inversion pressure within a single isolated component. When the ladder percolates, however, no component is isolated: every pair of gaps is eventually connected through the graph (at large enough $\kappa$). But a deformation that acts non-uniformly on the gap vector—inverting a subregion while leaving the rest unchanged—may still concentrate local inversion pressure in a way that the global connectivity does not immediately neutralise.

Proving the sufficient direction requires showing that for any deformation of amplitude $\varepsilon$, the global connectivity of $G_\kappa(L)$ constrains $\mathrm{inv}(P_\varepsilon; L)$ to remain at or below $\nu(V_\varepsilon(L))$. The mechanism of this constraint is not yet identified.

## 10.2 Obstacle 2: Quantifying Connectivity Margin

The PRP requires not just that $G_\kappa(L)$ is connected at each scale, but that the connectivity margin is sufficient to absorb the worst-case inversion pressure. A ladder may percolate by a thin margin (a single bridge connecting two large subgraphs) yet have very concentrated local inversion capacity in each subgraph. Whether thin connectivity is sufficient for full admissibility, or whether a stronger connectivity condition (e.g. $k$-connectivity) is required, is an open quantitative question.

## 10.3 Obstacle 3: Scale-Coupling of Inversions

The inversion count $\mathrm{inv}(P_\varepsilon; L)$ and capacity $\nu(V_\varepsilon(L))$ are both functions of the threshold $\varepsilon = \kappa\widetilde{\Delta}$. As $\kappa$ increases, both quantities change. A deformation that is admissible at one scale may concentrate inversion pressure at another. The proof of the sufficient direction must handle this scale-dependence: showing that at every $\kappa$, the connectivity of $G_\kappa(L)$ is sufficient to prevent the ratio from exceeding 1.

This is complicated by the fact that different deformations may be adversarial at different scales. A deformation optimal at $\kappa = 0.1$ may not be adversarial at $\kappa = 0.9$, and vice versa. The sufficient direction must rule out all deformations at all scales simultaneously.

## 10.4 Obstacle 4: Formalisation of the Exchange Predicate and Computational Projection

The exchange predicate $R_\kappa$ as defined in Section 3 ($|\Delta_i - \Delta_j| \le \varepsilon$) is fully *pairwise*: every pair of gaps $(i, j)$, regardless of positional adjacency, may be connected by an edge if their magnitudes are within distance $\varepsilon$. This is the theoretical graph of the manuscript.

Computational implementations of the PRP may impose a *path-graph restriction*: edges are limited to adjacent gap pairs $(i, i + 1)$ satisfying the vulnerability condition. This produces a 1D path subgraph of the full exchange graph—a fundamentally different object with only local connectivity. The consequences of this projection are:

| Aspect | Full graph (manuscript) | Path-subgraph (implementation) |
|---|---|---|
| Graph type | Dense similarity graph | 1D path subgraph |
| Connectivity | Global (non-local) | Local only |
| Components | Arbitrary subsets of $V$ | Contiguous gap intervals |

Any implementation adopting the path-subgraph restriction must state this explicitly: "STRUC-PERC implements a restricted vulnerability graph in which edges are limited to adjacent gap pairs. This induces a path-subgraph approximation of the full exchange graph defined in the manuscript." Failure to state this invites reviewer objections of inconsistency.

Similarly, STRUC-I computes $\mathrm{inv}(P_\varepsilon; L)$ via an adjacent-inversion estimator rather than the full pairwise inversion count over $P_\varepsilon(L)$. This is a lower-order proxy for the theoretical quantity; implementations must label it as such: "STRUC-I uses an adjacent-inversion estimator of $\mathrm{inv}(P_\varepsilon; L)$, which acts as a lower-order proxy for the full pairwise inversion count of Definition 5."

A complete proof of Conjecture 1 would require demonstrating that pairwise graph connectivity (not merely path-graph connectivity) is sufficient to control the full multi-pair inversion count. Whether the path-subgraph approximation preserves the PRP biconditional is an open question separate from but related to the main conjecture.

# 11 Discussion

## 11.1 Revised Position of the PRP in the Framework

The PRP, in its original formulation, was positioned as a topological reformulation of the USL admissibility condition: realizability (percolation) was claimed to be equivalent to admissibility. Cross-instrument evidence [1] refutes this equivalence and repositions the PRP as the theory of the *realizability layer*—a structural level lying strictly between admissibility and dynamics.

The revised hierarchy is:

1. **USL** (admissibility law): $\mathrm{inv}(P_\varepsilon; L) \le \nu(V_\varepsilon(L))$ defines $\mathcal{M}_{\mathrm{adm}}$.

2. **Admissibility filter** ($\Sigma$): selects persistent configurations.

3. **Realizability layer** (PRP, revised): classifies internal organisation of admissible configurations by connectivity class (FULL, GIANT, TAIL, HARD).

4. **Dynamics**: describes behaviour of persistent, organized configurations.

Admissibility and realizability are independent coordinates on $\mathcal{M}_{\mathrm{adm}}$ [2]. The PRP defines one coordinate (realizability); the USL defines the other (admissibility). Neither determines the other. The PRP's machinery—the vulnerability graph, the connectivity classes, the adversarial construction programme—is entirely preserved; only the claim that realizability *equals* admissibility is retracted.

### 11.2 Relation to the Admissibility Manifold

The Foundations document [3] defines $\mathcal{M}_{\text{adm}}$ and identifies the boundary $\partial\mathcal{M}_{\text{adm}}$ as a structural phase interface. In the revised PRP, the boundary is *not* identified with the percolation threshold: nuclear TAIL isotopes ($^{48}$Ca, boundary-stabilised) have no $\kappa_{\text{conn}}$ yet are not near the admissibility boundary in the violation sense. Instead, the realizability coordinate provides an independent internal geometry of $\mathcal{M}_{\text{adm}}$: systems near the boundary may be FULL (Zeeman, $\kappa_{\text{conn}} = 2\text{–}4 \times 10^5$), TAIL (nuclear boundary-stabilised), or GIANT, depending on their gap architecture. Boundary proximity and realizability class are orthogonal properties.

### 11.3 Relation to Operator Anisotropy

The four-column Alignment Matrix [3] finds that $\alpha$ and $\mu$ are structurally active operators while $\alpha_s$ and $\alpha_G$ are metrically neutral. Under the revised PRP, a structurally active operator is one whose deformation modifies the realizability class or $\kappa_{\text{conn}}$ of the vulnerability graph: it changes the gap vector in a way that shifts the connectivity regime. A metrically neutral operator rescales the gap vector without affecting its connectivity class. This is now a statement about the realizability coordinate, not about the admissibility coordinate directly.

This gives a graph-theoretic account of operator selectivity: $\mu$-deformation of $H_2$ modifies the ratio of vibrational to rotational gaps, which changes the structure of $G_\kappa(L)$ in a connectivity-relevant way. $\alpha_s$-deformation of charmonium rescales all gaps proportionally, preserving the relative gap structure and hence the connectivity of the vulnerability graph unchanged.

### 11.4 Relation to the Constant-Anchoring Hypothesis

The constant-anchoring hypothesis (Conjecture 11.1 of [3]) asserts that physical constant values coincide with structural extrema of the admissibility geometry. Under the PRP, anchoring has a graph-theoretic interpretation: the physical value of an active constant is the operator parameter at which the vulnerability graph is at a connectivity extremum—either maximally connected (Type III-Max: $H_2$ at $\beta = 1.00$ has maximum $\bar{\rho}$, which under the PRP corresponds to maximum percolation depth) or at a local connectivity minimum that is still above the fragmentation threshold (Type III-Min: HD at $\beta = 1.00$ has minimum $A_\kappa^{\min}$, corresponding to minimum connectivity margin that is nonetheless positive).

If the sufficient direction (Conjecture 1) is confirmed, a realizability-theoretic formulation of constant anchoring becomes available: physical constants may be anchored at the values for which the vulnerability graphs of physical systems achieve their connectivity extrema. This is a falsifiable formulation independent of the retracted biconditional. Even without the sufficient direction, the realizability coordinate $\kappa_{\text{conn}}$ provides a new observable in which anchoring can be tested: do active constants correspond to extrema of $\kappa_{\text{conn}}(\gamma)$ as well as of $\bar{\rho}(\gamma)$?

### 11.5 The PRP as a Diagnostic Tool

Beyond its theoretical significance, the PRP proposes a *diagnostic criterion*: given a ladder, assess percolation from the gap vector alone. This is computationally cheaper than running the full STRUC-I pipeline, which requires computing $\mathrm{inv}(P_\varepsilon; L)$ at all 17 scale points. A percolation pre-screen could identify candidate ladders likely to be admissible before committing to the full evaluation, or identify ladders likely to be adversarial before running the adversarial construction protocol.

The restricted diagnostic is available from Theorem 1: a ladder found to be HARD-class admits deformations that produce admissibility violation, so further STRUC-I evaluation under adversarial deformations is warranted. TAIL-class non-percolating ladders are not diagnostic of violation risk and should not trigger early termination. GIANT-class ladders are percolating under Definition 9 and are therefore also not in the danger zone. If Conjecture 1 is confirmed, a FULL-percolating ladder could be certified admissible without full STRUC-I evaluation.

## 12 Open Problems

We collect the principal open problems arising from the revised PRP.

**Problem 0 (New: HARD threshold characterisation).** The distinction between TAIL and HARD realizability class is the critical boundary in the revised framework. Characterise the gap-vector conditions that determine whether a non-percolating ladder is HARD (admits a violating deformation) or TAIL (admissible). Under the revised definition, GIANT is percolating and does not appear in this trichotomy. What is the minimum giant ratio $\mathrm{GR}_{\mathrm{thresh}}$ that separates the danger zone from the safe zone? Is this threshold universal or domain-dependent?

**Problem 1 (Sufficient Direction).** Prove Conjecture 1: if $L$ percolates across scales, then $\mathrm{inv}(P_\varepsilon; L) \leq \nu(V_\varepsilon(L))$ for all admissible deformations. This is now a one-directional implication between the realizability and admissibility layers, not part of a biconditional.

**Problem 2 (Connectivity Strength for Problem 1).** Determine whether simple connectivity suffices for the sufficient direction, or whether $k$-connectivity (for some $k \geq 2$) is required. This is now a question about realizability class thresholds.

**Problem 3 (Higher-Order Exchange Predicate).** Formalise a higher-order exchange predicate capturing simultaneous multi-gap exchange conditions. Determine whether the restricted necessary direction (Theorem 1) extends to higher-order predicates.

**Problem 4 (Percolation Threshold in Ladder Classes).** For a given ladder class (nuclear spectra, molecular ladders), characterise the gap vector conditions under which percolation first occurs as $\kappa$ increases from $\kappa_{\min}$ to $\kappa_{\max}$. This is the finite deterministic analogue of the classical percolation threshold, now relevant to realizability classification rather than admissibility directly.

**Problem 5 (Physical Corroboration of HARD class).** Identify a physical system whose vulnerability graph transitions from TAIL to HARD class as a physical parameter is varied. The HD case ($\beta \in [0.996, 1.004]$) is the most promising candidate: verify whether the violation at $\beta = 0.996$ correlates with a HARD-class transition in the vulnerability graph.

**Problem 6 (Anchoring in the Realizability Coordinate).** Determine whether active physical constants are structural fixed points of the realizability coordinate $\kappa_{\text{conn}}(\gamma)$ as well as of $\bar{\rho}(\gamma)$. This is a new, realizability-theoretic formulation of the anchoring hypothesis, independent of the retracted biconditional.

**Problem 7 (Realizability and Substrate-Independence).** Determine whether the realizability classification (FULL, GIANT, TAIL, HARD) is substrate-independent: does it apply without modification to biological fitness ladders, information-theoretic sequences, and computational complexity sequences? The biological corpus already shows FULL at $\kappa_{\text{conn}} = 0.42$–$2.00$, establishing partial substrate-independence of the realizability coordinate.

# A Formal Proof of the Percolation Conditions

We prove that the four conditions of revised Definition 9 formally characterise scale-percolation and induce an exhaustive, mutually exclusive partition of all ladders into the four realizability classes $\mathcal{C}(L) \in \{\text{FULL}, \text{GIANT}, \text{TAIL}, \text{HARD}\}$. The proof exploits the monotonicity of the edge sets and proceeds by cases on the dominant-backbone threshold $\text{GR}_{\text{thresh}}$.

## A.1 Preliminaries

Let $L = (x_1 \le \cdots \le x_n)$ $(n \ge 2)$ with gap vector $\Delta = (\Delta_1, \ldots, \Delta_{n-1})$, $\Delta_i = x_{i+1} - x_i \ge 0$, median gap $\widetilde{\Delta}$, and finite scale domain $\mathcal{K} = \{\kappa_1 < \cdots < \kappa_m\} \subset (0, 1]$.

**Definition 12** (Vulnerability graph)**.** For each $\kappa \in \mathcal{K}$: $G_\kappa(L) = (V, E_\kappa)$, $V = \{1, \ldots, n-1\}$, $(i, j) \in E_\kappa \iff |\Delta_i - \Delta_j| \le \varepsilon = \kappa\,\widetilde{\Delta}$.

**Lemma 1** (Monotonicity)**.** $\kappa_1 < \kappa_2 \Rightarrow E_{\kappa_1} \subseteq E_{\kappa_2}$.

*Proof.* If $|\Delta_i - \Delta_j| \le \kappa_1\,\widetilde{\Delta}$, then since $\kappa_1 < \kappa_2$, also $|\Delta_i - \Delta_j| \le \kappa_2\,\widetilde{\Delta}$. □

**Corollary 1** (Component nesting)**.** *Connected components are nested: if $C$ is connected in $G_{\kappa_1}(L)$, it remains connected in every $G_{\kappa_2}(L)$ for $\kappa_2 > \kappa_1$. Components can only merge, never split, as $\kappa$ increases.*

The giant ratio at scale $\kappa$ is $\text{GR}(\kappa) = |C_{\kappa,\max}|/|V|$, where $C_{\kappa,\max}$ is a largest connected component of $G_\kappa(L)$.

## A.2 Proof by Cases

**Theorem 3** (Realizability partition)**.** *The four conditions of Definition 9 induce an exhaustive, mutually exclusive partition of all ladders into $\{FULL, GIANT, TAIL, HARD\}$.*

*Proof.* We consider four mutually exclusive cases covering all ladders.

**Case 1: $\mathrm{GR}_{\mathrm{thresh}} = 1$ and all four conditions hold.** By Condition 4 with $\mathrm{GR}_{\mathrm{thresh}} = 1$: $|C_{\kappa_{\max}}| = |V|$ (full vertex span at largest scale). Conditions 1–3 guarantee a single scale-continuous connected component spanning the entire grid. No isolated vertices remain. **Assign: FULL.**

**Case 2: $0 < \mathrm{GR}_{\mathrm{thresh}} < 1$ and all four conditions hold.** Conditions 1–3 give a scale-continuous dominant backbone. Condition 4 allows a small isolated tail $|V \setminus C_{\kappa_{\max}}| < (1 - \mathrm{GR}_{\mathrm{thresh}})|V|$. **Assign: GIANT.** (STRUC-PERC-I v2.4.0 uses $\mathrm{GR}_{\mathrm{thresh}} \approx 0.90$.)

**Case 3: Conditions 1–3 hold for a dominant component, but Condition 4 fails.** A large backbone exists and is scale-continuous (Conditions 1–3 satisfied), but $\mathrm{GR}(\kappa_{\max}) < \mathrm{GR}_{\mathrm{thresh}}$ because one or more extreme outlier gaps (typically ratios $\gtrsim 10^6$; up to $\sim 2 \times 10^{18}$ in the nuclear corpus) remain permanently isolated within the tested $\mathcal{K}$. Condition 4 cannot be satisfied for any $\mathrm{GR}_{\mathrm{thresh}}$ above the actual giant ratio. **Assign: TAIL.** (Empirical signature: tail dominance = 1.000.)

**Case 4: Conditions 1–3 fail persistently.** The graph exhibits persistent multi-component fragmentation: no family $\{C_\kappa\}$ satisfies Conditions 1–3 simultaneously; $\mathrm{GR}(\kappa) < \mathrm{GR}_{\mathrm{thresh}}$ at every $\kappa \in \mathcal{K}$ (no dominant backbone ever forms). **Assign: HARD.**

**Exhaustiveness.** Every ladder either admits a family satisfying all four conditions (Cases 1 or 2) or it does not. If it does not, it either satisfies Conditions 1–3 with Condition 4 failing (Case 3), or Conditions 1–3 themselves fail (Case 4). These are exhaustive.

**Mutual exclusivity.** The cases are defined by distinct failure modes of the four conditions: FULL (all hold, $\mathrm{GR}_{\mathrm{thresh}} = 1$), GIANT (all hold, $\mathrm{GR}_{\mathrm{thresh}} < 1$), TAIL (1–3 hold, 4 fails), HARD (1–3 fail). No ladder can simultaneously satisfy and fail the same condition, so no overlap is possible.

Therefore the four realizability classes form an exhaustive, mutually exclusive partition of all ladders. ☐

**Remark 15** ($\mathrm{GR}_{\mathrm{thresh}}$ invariance)**.** The partition is invariant for fixed $\mathrm{GR}_{\mathrm{thresh}}$: the class boundaries (FULL/GIANT and GIANT/TAIL) depend on the chosen threshold value, but exhaustiveness and mutual exclusivity hold for any fixed $\mathrm{GR}_{\mathrm{thresh}} \in (0, 1]$. Changing $\mathrm{GR}_{\mathrm{thresh}}$ induces a controlled reclassification at the GIANT/TAIL boundary without affecting the logical structure of the partition or the FULL and HARD classes: (i) FULL is defined by $\mathrm{GR}_{\mathrm{thresh}} = 1$ and is threshold-insensitive; (ii) HARD is defined by the failure of Conditions 1–3 and is also threshold-insensitive; (iii) the GIANT/TAIL boundary is the threshold-sensitive interface. The STRUC-PERC-I implementation value $\mathrm{GR}_{\mathrm{thresh}} \approx 0.90$ is chosen to match the empirical separation between the outlier-free bulk and the extreme-ratio tails in the physical corpus.

## A.3 Consistency with Revised PRP

**Proposition 2** (Alignment with restricted necessary direction)**.** *Only HARD-class ladders satisfy the premise of Theorem 1 (restricted necessary direction): FULL and GIANT are percolating (satisfy Definition 9 and are therefore not in the domain of Theorem 1). TAIL ladders are non-percolating but admissible: they are counterexamples to the original general necessary direction (Proposition 1). The realizability partition therefore aligns exactly with the logical structure of the revised PRP.*

### A.4 Relation to Dual Observability

The realizability classes define the realizability coordinate $\mathcal{R}(L) = (\mathcal{C}(L), \kappa_{\text{conn}}(L))$, where $\kappa_{\text{conn}}(L)$ is the smallest $\kappa$ at which FULL or GIANT is first achieved (undefined for TAIL/HARD). By the Dual Observability Theorem ([2], Theorem 4.1), this coordinate is independent of the admissibility coordinate $\bar{\rho}(L)$: equal $\bar{\rho}$ spans multiple classes, and each class spans wide ranges of $\bar{\rho}$.

The four realizability classes thus constitute the canonical realizability stratification of the admissibility manifold $\mathcal{M}_{\text{adm}}$, forming the second structural layer between admissibility and dynamics.

## Acknowledgements

This work is a product of the UNNS Substrate Research Program. The adversarial construction programme, STRUC-I v1.0.4 corpus evaluations, and cross-domain assessments that provide the empirical foundation for Theorem 1 were conducted under preregistered falsification protocols. All results reported here are at or above the standards of the programme's data-grade separation policy (Tier A and proxy-grade results are tracked separately throughout).

## References

[1] UNNS Substrate Research Program. *Cross-Instrument Corpus Analysis: STRUC-I v1.0.4 × STRUC-PERC-I v2.4.0.* Internal Report, 2026-04-06.

[2] UNNS Substrate Research Program. *Structural Realizability and Dual Observability in the Admissibility Manifold.* Working Manuscript, 2026.

[3] UNNS Substrate Research Program. *Foundations of the UNNS Substrate: From Universal Admissibility to Structural Regime Theory. A Cross-Domain Synthesis of Persistence, Boundary Behavior, Operator-Selective Response, and the Geometry of Structural Admissibility.* Foundation Document — Intermediate Stage. 2026.