Perturbation-Admissible Structural Stability in the UNNS Substrate

Admissibility Geometry, Phase Boundaries, and Quotient Descent in Co-Seismic Rank and Topology Invariance

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Abstract

Within the UNNS Substrate framework, every co-seismic displacement hierarchy is a realization inside an admissible operator family, and structural stability is the statement that the induced signature descends through the quotient by admissible transforms. We formalize this descent condition as perturbation-admissible structural stability, instantiated concretely on the LXV seismic chamber suite. Smoothing windows $w \in W$ act as nested resolution morphisms on the substrate; the perturbation envelope $\|\mathbf{u}_w - \mathbf{u}_1\| \leq \sigma_P$ is bounded admissible deformation within the operator grammar. We introduce two rigidity moduli – the magnitude modulus $\mathcal{R}_{\text{mag}} = \min_k \Delta_k/(2\sigma_P)$ and the geometric modulus $\mathcal{R}_{\text{geo}} = \Theta_{\min}/(2\delta_P)$ – whose joint value $\boldsymbol{\mathcal{R}} = (\mathcal{R}_{\text{mag}}, \mathcal{R}_{\text{geo}})$ constitutes the admissibility margin of the event. We prove that $\boldsymbol{\mathcal{R}} > (1,1)$ is sufficient for full descent (Order Rigidity and Directional Stability), derive tight inversion bounds via the matching number $\nu(V)$ of vulnerable gaps, and establish a Phase-Boundary Theorem showing that boundary degeneracy is structurally forced whenever either modulus reaches 1. Three earthquake events spanning two orders of magnitude in $\mathcal{R}_{\text{mag}}$ confirm the predicted phase structure and demonstrate that the admissibility margin encodes near-fault geometry independently of event magnitude.

1 Preliminaries

Let $\mathcal{N}$ be a finite station set with $|\mathcal{N}| = N$.

Let $S_w(i)$ denote the co-seismic horizontal displacement magnitude computed under smoothing window w.

Let R_w denote the total order induced by S_w.

Let $w_0 = 1$ denote the baseline window throughout.

Definition 1 (Inversion Distance). *For window w, define*

$$D(w) = \#\{(i,j) : R_{w_0}(i) < R_{w_0}(j) \text{ but } R_w(i) > R_w(j)\}.$$

Definition 2 (Perturbation-Admissible Family). *A smoothing family $\mathcal{W}$ is perturbation-admissible if:*

1. (Symmetry) *All windows are symmetric and centered. Used in: the sub-Gaussian proxy bound of Lemma 1, where centered averaging implies the smoothed noise has zero mean.*
2. (Guard consistency) *Guard convention is fixed across all windows. Used in: the epoch count m is the same for every w, so the envelope $\sigma_P(\alpha)$ is computed uniformly.*
3. (Detrending invariance) *Detrending protocol is invariant. Used in: the residual model of Definition 7, which assumes post-detrending mean-zero noise.*
4. (Finiteness) *Window size is finite. Used in: all probability bounds require finitely many samples $m < \infty$.*

2 Rigid–Nonrigid Transition Principle (RNP)

Principle 1 (Rigid–Nonrigid Transition). *Let $\mathcal{O}$ be an induced order under a perturbation operator P_ϵ. There exists a critical perturbation scale ϵ_c such that:*

$$\epsilon < \epsilon_c \implies \mathcal{O} \text{ rigid}$$
$$\epsilon \geq \epsilon_c \implies \mathcal{O} \text{ admits degeneracy}$$

Seismic instantiation. In the co-seismic ranking context the RNP critical scale is instantiated as

$$\epsilon_c = \tfrac{1}{2} \min_k \Delta_k,$$

the half-width of the smallest adjacent gap. The perturbation amplitude is σ_P (see Definition 4 and 5). The Rigidity Modulus $\mathcal{R} = \epsilon_c/\sigma_P$ therefore measures the distance from the transition boundary: $\mathcal{R} > 1$ is exactly the RNP condition $\sigma_P < \epsilon_c$, while $\mathcal{R} \leq 1$ places the system at or below the transition threshold.

3 Rigidity Modulus

Definition 3 (Adjacent Gap). *For baseline ranking R_{w_0} with magnitudes*

$$S_{w_0}(i_1) \geq S_{w_0}(i_2) \geq \cdots \geq S_{w_0}(i_N),$$

define adjacent gaps:

$$\Delta_k = S_{w_0}(i_k) - S_{w_0}(i_{k+1}).$$

The Rigidity Modulus depends on how σ_P is estimated. We distinguish three notions that share intuition but differ operationally; all are defined relative to the same canonical form $\mathcal{R} = \min_k \Delta_k/(2\sigma_P)$.

Definition 4 (Empirical Perturbation Amplitude). *The* empirical *perturbation amplitude is the worst-case observed deviation:*

$$\sigma_P^{(\mathrm{emp})} := \max_{s\in\mathcal{N},\, w\in W\setminus\{w_0\}} \big|S_w(s) - S_{w_0}(s)\big|.$$

Definition 5 (Probabilistic Perturbation Envelope). *For confidence level $\alpha \in (0,1)$, the* probabilistic *perturbation envelope is*

$$\sigma_P(\alpha) := \max_{w\in W} \left(2c_{\mathrm{med}}\, \sigma \sqrt{1+\tfrac{1}{w}} \sqrt{\tfrac{\log(4/\alpha)}{m}}\right),$$

where σ is the sub-Gaussian noise proxy (Section 5), m is the number of usable epochs (Definition 7), and c_{med} is a universal constant bounded in Remark 2.

Remark 1 (Relationship between the two amplitudes). *By Theorem 2, $\sigma_P^{(\mathrm{emp})} \leq \sigma_P(\alpha)$ with probability at least $1-\alpha$. Hence the empirical Rigidity Modulus*

$$\widehat{\mathcal{R}}(\alpha) := \frac{\min_k \Delta_k}{2\,\sigma_P(\alpha)}$$

is a data-specific lower bound *on the true modulus: if $\widehat{\mathcal{R}}(\alpha) > 1$ then the rank is rigid with probability at least $1-\alpha$.*

Definition 6 (Rigidity Modulus – Canonical Form). *For either amplitude variant:*

$$\mathcal{R} := \frac{\min_k \Delta_k}{2\sigma_P}.$$

Running example. For LXV-B2 (El Mayor 2010), baseline magnitudes are $104.94, 49.11, 7.95, 7.46, 6.96$ mm, giving $\min \Delta_k = 0.484$ mm. Empirically $\sigma_P^{(\text{emp})} \approx 1.3$ mm, so $\mathcal{R} = 0.484/(2 \times 1.3) \approx 0.19$. Vulnerable gaps: $|V| = 2$ (the two sub-millimetre tail gaps). Degeneracy Bound: $D(w) \leq |V|(N-1) = 2 \times 4 = 8$. Observed $D = 1$ in selected windows – consistent with boundary degeneracy. This worked example is revisited at each theoretical step below.

4 Deterministic Order Rigidity

Theorem 1 (Order Rigidity). *If* $\mathcal{R} > 1$ *(equivalently,* $\min_k \Delta_k > 2\sigma_P$*), then*

$$D(w) = 0 \quad \forall w \in \mathcal{W}.$$

Contradiction outline. Assume $\mathcal{R} > 1$ but $D(w) > 0$ for some w. Then there exists an adjacent pair (i_k, i_{k+1}) such that $S_w(i_k) < S_w(i_{k+1})$. By the perturbation bound each station's displacement changes by at most σ_P (Admissibility condition 1), so

$$S_w(i_k) \geq S_{w_0}(i_k) - \sigma_P, \quad S_w(i_{k+1}) \leq S_{w_0}(i_{k+1}) + \sigma_P.$$

Inversion requires $S_w(i_{k+1}) - S_w(i_k) > 0$, hence $\Delta_k = S_{w_0}(i_k) - S_{w_0}(i_{k+1}) \leq 2\sigma_P$, contradicting $\mathcal{R} > 1$. □

Running example (continued). LXV-B2 has $\mathcal{R} \approx 0.19 < 1$, so Theorem 1 does *not* apply; inversions are permitted. LXV-A (Kumamoto) has $\mathcal{R} \approx 21.6 \gg 1$, so the theorem guarantees $D(w) = 0$ for every window – as observed.

5 Probabilistic Quantitative Rigidity

Theorem 1 is the worst-case (deterministic) result. We now derive a stronger probabilistic version that makes $\mathcal{R}$ computable from noise observations.

5.1 Assumptions

Definition 7 (Residual Noise Model). *After the LXV pre-fit detrending step, suppose each component residual* $\varepsilon_s(t)$ *(for each station* s*) is conditionally mean-zero and* σ*-sub-Gaussian:*

$$\mathbb{E}[\exp(\lambda \varepsilon_s(t))] \leq \exp\left(\frac{\lambda^2 \sigma^2}{2}\right) \quad \forall \lambda \in \mathbb{R}.$$

We assume independence across epochs within each side-window (pre vs. post) for bounding purposes. Let m_{pre} and m_{post} be the number of usable epochs in each side after guard and coverage gates; set $m := \min(m_{\text{pre}}, m_{\text{post}})$.

5.2 Centered Moving-Average Smoothing Bound

Let MA_w denote centered moving average of odd window length w. For i.i.d. mean-zero σ-sub-Gaussian noise, the smoothed residual $\tilde{\varepsilon}^{(w)}(t) := (\text{MA}_w \varepsilon)(t)$ is sub-Gaussian with proxy $\sigma_w \le \sigma/\sqrt{w}$ (Admissibility condition 1 is used here). Consequently, the difference between baseline and smoothed noise satisfies

$$\varepsilon(t) - \tilde{\varepsilon}^{(w)}(t) \text{ is sub-Gaussian with proxy } \sigma_\Delta(w) \le \sigma\sqrt{1 + \tfrac{1}{w}}.$$

5.3 Median-of-Window Perturbation Bound

LXV step extraction uses guarded window summaries on the pre and post sides. Let $\text{Med}_m(\cdot)$ denote the sample median of m independent samples.

Lemma 1 (Median Concentration). *There exists a universal constant $c_{\text{med}} > 0$ such that for any $\alpha \in (0, 1)$,*

$$\Pr\left(|\text{Med}_m(X) - \text{Med}(X)| \ge c_{\text{med}}\, \sigma_X \sqrt{\frac{\log(2/\alpha)}{m}} \right) \le \alpha,$$

whenever X is σ_X-sub-Gaussian.

Remark 2 (Bound on c_{med}). *For Gaussian noise, standard order statistics give $c_{\text{med}} = \sqrt{\pi/2} \approx 1.253$. A conservative choice valid for any sub-Gaussian distribution is $c_{\text{med}} = 2$, which is sufficient for all empirical computations in Section 12.*

Theorem 2 (Explicit Smoothing Envelope for Step Estimates). *Fix $\alpha \in (0, 1)$ and $w \in W$. With probability at least $1 - \alpha$,*

$$\left| \widehat{\Delta}_s(w) - \widehat{\Delta}_s(1) \right| \;\le\; 2c_{\text{med}}\, \sigma \sqrt{1 + \tfrac{1}{w}} \sqrt{\tfrac{\log(4/\alpha)}{m}} \;\le\; \sigma_P(\alpha).$$

5.4 Probabilistic Rigidity

Theorem 3 (Quantitative Order Rigidity). *If $\widehat{\mathcal{R}}(\alpha) > 1$ then with probability at least $1 - \alpha$,*

$$D(w) = 0 \quad \forall w \in W.$$

Proof sketch. If $\widehat{\mathcal{R}}(\alpha) > 1$, then $\min_k \Delta_k > 2\sigma_P(\alpha)$. By Theorem 2, with probability at least $1-\alpha$ every station deviation satisfies $|S_w(s) - S_1(s)| \leq \sigma_P(\alpha)$. Applying the deterministic argument of Theorem 1 with $\sigma_P = \sigma_P(\alpha)$ completes the proof. □

Note that Theorem 1 is the limiting case $\alpha \to 0$ (worst-case envelope) of Theorem 3: the deterministic result is the boundary of the probabilistic family.

5.5 Operational estimation of σ

The parameter σ is estimable from detrended residuals. A robust estimator is the scaled median absolute deviation:

$$\widehat{\sigma} := 1.4826 \cdot \mathrm{MAD}(\varepsilon_s(t)).$$

This yields an empirical $\widehat{\sigma}_P(\alpha)$ and therefore an empirical $\widehat{\mathcal{R}}(\alpha)$.

Remark 3 (SNR gate as a noise-model check)**.** *The LXV chambers gate each station on* $\mathrm{SNR}_h \geq 5$*, where*

$$\mathrm{SNR}_h = \frac{S_1(s)}{1.4826 \cdot \mathrm{MAD}(\varepsilon_s(t))} = \frac{S_1(s)}{\widehat{\sigma}_s}.$$

This is exactly the condition $S_1(s) \geq 5\widehat{\sigma}_s$: the baseline signal must exceed five noise standard deviations. Stations failing this gate have unreliable Δ_k estimates and are excluded before computing $\mathcal{R}$.

6 Boundary Regime and Degeneracy Index

Definition 8 (Degenerate Boundary)**.** *If $0 < \mathcal{R} \leq 1$, bounded inversion may occur. Define the* vulnerable gap set

$$V = \{k \in \{1, \ldots, N-1\} : \Delta_k \leq 2\sigma_P\}.$$

and the Degeneracy Index

$$\mathcal{D} = \frac{|V|}{N-1}.$$

Theorem 4 (General Degeneracy Bound)**.** *Fix $w_0 = 1$ and let baseline magnitudes be ordered $S_1(i_1) \geq \cdots \geq S_1(i_N)$ with adjacent gaps Δ_k. Assume* *a uniform perturbation envelope* σ_P *such that* $|S_w(s) - S_1(s)| \le \sigma_P$ *for every* $s \in \mathcal{N}$*,* $w \in W$*. Then for every* $w \in W$*:*

$$D(w) \;\le\; \sum_{k\in V}(N-k) \;\le\; |V|\,(N-1).$$

In particular, if $V = \varnothing$ *(equivalently* $\mathcal{R} > 1$*) then* $D(w) = 0$ *for all* $w \in W$*.*

Proof sketch. Every inverted pair (i_a, i_b) with $a < b$ can be charged to a vulnerable adjacent gap in V lying between positions a and b: by the envelope bound applied twice, $\sum_{k=m}^{b-1} \Delta_k \le 2\sigma_P$, so some $k \in V$ must exist in that block. Counting all pairs that can be charged to a given $k \in V$ yields at most $N - k$ choices for the upper index, giving the sum $\sum_{k\in V}(N - k)$. The crude bound $N - k \le N - 1$ produces the second inequality. ☐

Theorem 5 (Degeneracy Bound – Adjacent-Swap Regime)**.** *Under the same hypotheses as Theorem 4, assume additionally the* adjacent-swap regime*:*

$$\max_{s\in\mathcal{N}} |r_w(s) - r_1(s)| \le 1,$$

i.e. no station shifts by more than one rank relative to baseline. Then for every $w \in W$*,*

$$D(w) \le |V|.$$

Proof sketch. Under $\max_s |r_w(s) - r_1(s)| \le 1$, every inversion must be an *adjacent* swap across some boundary between positions k and $k + 1$ in the baseline ordering. Such a swap can occur only if $\Delta_k \le 2\sigma_P$, hence it must be charged to some $k \in V$. Under $\max_s |r_w(s) - r_1(s)| \le 1$, crossing boundary k can only occur via the single adjacent transposition of (i_k, i_{k+1}); any additional inversion involving that boundary would force at least one element to shift by two ranks, violating the regime assumption. Therefore distinct inversions correspond to distinct elements of V, giving $D(w) \le |V|$. ☐

Running example (continued). For LXV-B2: $N = 5$, $|V| = 2$. Under the general bound (Theorem 4): $D(w) \le 2\times 4 = 8$. Under the adjacent-swap regime of Theorem 5, which the chamber enforces via `max_rank_shift` ≤ 1: $D(w) \le |V| = 2$. The chamber's operational budget $k_{\text{allowed}} = 1$ is thus conservative relative to the tighter bound of 2.

Theorem 6 (Degeneracy Index Bounds the Maximum Inversion Budget)**.** *Fix baseline window* $w_0 = 1$ *and let the baseline magnitudes be ordered* $S_1(i_1) \ge \cdots \ge S_1(i_N)$ *with adjacent gaps* Δ_k*. Assume a uniform perturbation envelope* σ_P *such that* $|S_w(s) - S_1(s)| \le \sigma_P$ *for all* $s \in \mathcal{N}$*,* $w \in W$*.*

Define the vulnerable-gap set $V = \{k : \Delta_k \le 2\sigma_P\}$ *and Degeneracy Index* $\mathcal{D} = |V|/(N-1)$. *Assume the adjacent-swap regime* $\max_s |r_w(s) - r_1(s)| \le 1$. *Then for every* $w \in W$,

$$D(w) \le |V| = (N-1)\mathcal{D}.$$

Consequently, any inversion-budget rule $D(w) \le k_{\text{allowed}}$ *is* guaranteed achievable *under the adjacent-swap regime provided* $k_{\text{allowed}} \ge (N-1)\mathcal{D}$. *In particular, choosing* $k_{\text{allowed}} := |V|$ *is a principled* conservative upper budget. *The minimal necessary budget in the adjacent-swap regime is* $k^{\min}_{\text{allowed}} = \nu(V)$ *(Theorem 7).*

Proof. Fix $w \in W$. Under the adjacent-swap regime, every inversion is a swap of two items adjacent in the baseline order, crossing exactly one boundary k. At a crossing of boundary k: $S_1(i_k) \ge S_1(i_{k+1})$ but $S_w(i_k) < S_w(i_{k+1})$. Applying the envelope twice,

$$\Delta_k = S_1(i_k) - S_1(i_{k+1}) \le (S_1(i_k) - S_w(i_k)) + (S_w(i_{k+1}) - S_1(i_{k+1})) \le 2\sigma_P,$$

hence $k \in V$. Moreover, since no station moves more than one rank, no single boundary k can be crossed by more than one independent inversion: doing so would require a rank shift of magnitude ≥ 2. Therefore distinct inversions correspond to distinct elements of V, giving $D(w) \le |V| = (N-1)\mathcal{D}$. The budget implications are immediate. □

Corollary 1 (Principled k_{allowed} Choice for LXV-B2)**.** *In LXV-B2, the gate* $\max_s |r_w(s) - r_1(s)| \le 1$ *is enforced. Therefore it is principled to set*

$$k_{\text{allowed}} := |V|,$$

where $V = \{k : \Delta_k \le 2\sigma_P\}$ *is computed from baseline gaps and the smoothing envelope. The chamber's choice of* $k_{\text{allowed}} = 1$ *is conservative relative to the theoretically derived budget* $|V| = 2$.

Definition 9 (Matching Number of Vulnerable Gaps)**.** *Given envelope* σ_P *and vulnerable set* $V = \{k \in \{1, \dots, N-1\} : \Delta_k \le 2\sigma_P\}$, *a subset* $M \subseteq V$ *is* disjoint *if it contains no two adjacent indices:*

$$k \in M \implies (k-1) \notin M \text{ and } (k+1) \notin M.$$

The matching number *is*

$$\nu(V) := \max\{|M| : M \subseteq V \text{ is disjoint}\}.$$

Note that $\nu(V) \le |V|$, *with equality if and only if* V *contains no two adjacent indices.*

Remark 4 (One-line computation of $\nu(V)$ on a path)**.** *Sort V in increasing order and scan left-to-right; greedily pick index k whenever $k >$ lastPicked$+1$ (initialise lastPicked $= -\infty$). The count of picked indices equals $\nu(V)$. This is the standard maximum matching on a path graph, solvable in $O(|V|)$ time.*

Theorem 7 (Minimal Necessary Inversion Budget – Adjacent-Swap Regime)**.** *Assume the uniform envelope $|S_w(s) - S_1(s)| \leq \sigma_P$ for all $s \in \mathcal{N}$, $w \in W$, and the adjacent-swap regime $\max_s |r_w(s) - r_1(s)| \leq 1$.*

Then for every $w \in W$:

$$D(w) \leq \nu(V).$$

Moreover the bound is tight: there exists a perturbation satisfying both the envelope and the adjacent-swap regime for which $D(w) = \nu(V)$.

Consequently, the minimal certification budget that is guaranteed not to false-fail any admissible perturbation is

$$k_{\text{allowed}}^{\min} = \nu(V).$$

Proof sketch. (*Upper bound.*) Under $\max_s |r_w(s) - r_1(s)| \leq 1$, the permutation from R_1 to R_w is a product of *disjoint* adjacent transpositions: no element participates in two swaps, so crossed boundaries form a disjoint subset of $\{1, \ldots, N-1\}$. The envelope argument shows each crossed boundary k must satisfy $\Delta_k \leq 2\sigma_P$, hence lie in V. Therefore the crossed boundaries form a disjoint subset of V, and $D(w) \leq \nu(V)$.

(*Tightness.*) Let $M \subseteq V$ be a maximum disjoint subset, $|M| = \nu(V)$. For each $k \in M$, set $S_w(i_k) = S_1(i_k) - \sigma_P$ and $S_w(i_{k+1}) = S_1(i_{k+1}) + \sigma_P$. Since boundaries in M are disjoint, no element is perturbed twice; the rank shift of every element is at most 1, preserving the adjacent-swap regime. Each boundary $k \in M$ satisfies $\Delta_k \leq 2\sigma_P$, so $S_w(i_k) < S_w(i_{k+1})$, realising one inversion per boundary. Thus $D(w) = \nu(V)$.

(*Minimality.*) The tightness construction is an admissible instance, so any rule with $k_{\text{allowed}} < \nu(V)$ would false-fail it. Hence $k_{\text{allowed}}^{\min} = \nu(V)$. □

Corollary 2 (When $|V|$ is minimal)**.** *If V contains no two adjacent indices (all vulnerable gaps are separated), then $\nu(V) = |V|$ and the minimal necessary budget equals $|V|$.*

Remark 5 (LXV-B2 – El Mayor instantiation)**.** *El Mayor B2 has two vulnerable tail gaps at positions $k = 3$ and $k = 4$ (gaps of* 0.484 *mm and* 0.499 *mm respectively), which are* adjacent *indices. Therefore $|V| = 2$ but $\nu(V) = 1$: only one of the two boundaries can be swapped without forcing* *any element to shift two ranks. The minimal necessary inversion budget is $k^{\min}_{\text{allowed}} = \nu(V) = 1$, which equals the chamber's operational choice exactly. The chamber's gate is therefore not merely conservative – it is the* tight minimal budget *derived from the perturbation envelope and the adjacent-swap constraint.*

7 Inversion Distance and Rank Correlation

Lemma 2 (Kendall τ is determined by inversion count)**.** *Let R_1 and R_w be two total orders on N items with inversion count $D(w)$. Then Kendall's τ satisfies*

$$\tau(w) = 1 - \frac{4D(w)}{N(N-1)}.$$

Corollary 3 (Theorem-grade τ lower bounds from vulnerability)**.** *Under the assumptions of Theorem 4, for every $w \in W$:*

$$\tau(w) \ \geq \ 1 - \frac{4}{N(N-1)} \sum_{k \in V} (N-k) \ \geq \ 1 - \frac{4|V|}{N}.$$

If moreover the adjacent-swap regime of Theorem 5 holds, then

$$\tau(w) \ \geq \ 1 - \frac{4|V|}{N(N-1)}.$$

Definition 10 (Adjacent-swap benchmark model for Spearman ρ)**.** *Fix N and integer $k \geq 0$. The* benchmark model *takes R_w to be obtained from R_1 by performing exactly k disjoint adjacent swaps (each swap exchanges positions j and $j+1$ for some j, with all swapped pairs disjoint). Let $\rho_{\text{bench}}(N,k)$ denote Spearman's ρ between R_1 and R_w under this model.*

Lemma 3 (Spearman benchmark value under k disjoint adjacent swaps)**.** *Under the benchmark model of Definition 10,*

$$\rho_{\text{bench}}(N,k) = 1 - \frac{12k}{N(N^2-1)}.$$

Remark 6 (Status of ρ in this paper)**.** *Unlike Kendall τ, Spearman ρ is not determined solely by inversion count: different patterns of inversions yield different values of $\sum d_i^2$. Accordingly, ρ_{bench} is used in this paper as a* reporting benchmark *calibrated to the adjacent-swap interpretation (one tolerates at most $k = |V|$ such swaps), not as a theorem-grade lower bound valid for all perturbation patterns.*

Running example (LXV-B2, $N = 5$). The certification gate enforces $D(w) \le k_{\text{allowed}} = 1$. If a window yields exactly one inversion ($D(w) = 1$), then by Lemma 2,

$$\tau(w) = 1 - \frac{4}{5 \cdot 4} = 0.80.$$

Under the adjacent-swap benchmark model,

$$\rho_{\text{bench}}(5, 1) = 1 - \frac{12}{5(25 - 1)} = 0.90.$$

These match the reported minima in LXV-B2 (`kendall_min: 0.8`, `spearman_min: 0.9`). Kendall τ is fully determined by D, while Spearman ρ is reported relative to the adjacent-swap benchmark.

8 Topology Stability

Let $\theta_w(i)$ denote displacement azimuth at station i under window w. Let C_w be the clustering assignment under directional k-means.

Definition 11 (Angular Separation and Drift). *Let Θ denote the centroid separation between clusters (computed at baseline w_0). The* angular drift δ_P *induced by smoothing is bounded by the displacement perturbation normalised by the minimum baseline magnitude:*

$$\delta_P \le \frac{\sigma_P^{(\text{emp})}}{\min_s S_1(s)}.$$

This bound follows because the azimuth of a vector $\mathbf{v}$ changes by at most $\sin^{-1}(\delta/\|\mathbf{v}\|)$ under a perturbation of size δ; for small $\delta/\|\mathbf{v}\|$ this is bounded above by $\delta/\|\mathbf{v}\|$.

Theorem 8 (Topology Rigidity). *If*

$$\Theta > 2\delta_P,$$

then cluster assignment is invariant across all admissible windows:

$$\text{ARI}(C_w, C_{w'}) = 1 \quad \forall w, w'.$$

Remark 7 (TOPO_SINGLE edge case). *For events classified as `TOPO_SINGLE` (a single displacement cluster, e.g. Ridgecrest LXV-C2 with `sep` $= 0$), the centroid separation Θ is undefined because there is no second centroid. In this case* ARI $= 1$ *holds trivially and is not a validation of Theorem 8; it is a degenerate case the theorem does not cover. A single-cluster event provides no discriminating evidence for or against the topology framework.*

9 Perturbation Geometry and Directional Stability

9.1 Vector Representation of Station Displacements

For each station $s \in \mathcal{N}$, let the co-seismic displacement vector be

$$\mathbf{u}_w(s) = (d_E(s,w),\, d_N(s,w)) \in \mathbb{R}^2.$$

Define the baseline configuration $\mathbf{u}_1(s) := \mathbf{u}_{w_0}(s)$ with $w_0 = 1$, and let $\|\mathbf{u}\| := \sqrt{d_E^2 + d_N^2}$ denote Euclidean magnitude. We interpret smoothing as a perturbation operator

$$\mathbf{u}_w(s) = \mathbf{u}_1(s) + \boldsymbol{\varepsilon}_w(s),$$

with bounded perturbation envelope $\|\boldsymbol{\varepsilon}_w(s)\| \leq \sigma_P$ (Admissibility condition 1 ensures the envelope is uniform across stations and windows).

Remark 8 (Consistency with scalar envelope)**.** *The scalar perturbation used in Section 3 is* $|S_w(s) - S_1(s)| = \big|\|\mathbf{u}_w(s)\| - \|\mathbf{u}_1(s)\|\big| \leq \|\boldsymbol{\varepsilon}_w(s)\| \leq \sigma_P$ *by the reverse triangle inequality. The vector envelope thus implies the scalar envelope used in all rank-stability theorems.*

9.2 Angular Perturbation Envelope

Let $\theta_w(s) := \arg(\mathbf{u}_w(s))$ denote the displacement azimuth under window w. Define the angular perturbation bound

$$\delta_P := \sup_{s\in\mathcal{N},\, w\in W} |\theta_w(s) - \theta_1(s)|.$$

Lemma 4 (Angular Drift Bound)**.** *If* $\|\boldsymbol{\varepsilon}_w(s)\| \leq \sigma_P$ *and* $\|\mathbf{u}_1(s)\| \geq m > 0$, *then*

$$|\theta_w(s) - \theta_1(s)| \;\leq\; \arcsin\!\left(\frac{\sigma_P}{m}\right).$$

In particular,

$$\delta_P \;\leq\; \arcsin\!\left(\frac{\sigma_P}{\min_s \|\mathbf{u}_1(s)\|}\right) \;\leq\; \frac{\sigma_P}{\min_s \|\mathbf{u}_1(s)\|}$$

for small $\sigma_P / \min_s \|\mathbf{u}_1(s)\|$.

Proof. The maximal angular deviation of $\mathbf{u}_w = \mathbf{u}_1 + \boldsymbol{\varepsilon}$ relative to $\mathbf{u}_1$ is achieved when $\boldsymbol{\varepsilon}$ is orthogonal to $\mathbf{u}_1$. In that configuration,

$$\sin|\theta_w - \theta_1| = \frac{\|\boldsymbol{\varepsilon}\|}{\|\mathbf{u}_1\|} \leq \frac{\sigma_P}{\|\mathbf{u}_1(s)\|}.$$

Taking the supremum over s and w and applying arcsin completes the proof. $\square$

Remark 9 (Consistency with the δ_P bound in Section 8)**.** *Lemma 4 provides the rigorous derivation of the bound $\delta_P \le \sigma_P / \min_s S_1(s)$ stated in Section 8. The two are consistent: $S_1(s) = \|\mathbf{u}_1(s)\|$ by definition.*

9.3 Directional Cluster Separation

Let the baseline directional configuration partition the stations into K clusters $\mathcal{C}_1, \ldots, \mathcal{C}_K$ under directional k-means, with minimal inter-cluster angular separation

$$\Theta_{\min} = \min_{i \neq j} \inf_{s \in \mathcal{C}_i, t \in \mathcal{C}_j} |\theta_1(s) - \theta_1(t)|.$$

Theorem 9 (Directional Stability Theorem)**.** *If $\Theta_{\min} > 2\delta_P$, then for all $w \in W$ the cluster assignment is invariant:*

$$\mathcal{C}_i(w) = \mathcal{C}_i(1) \quad \forall i.$$

Proof. For any $s \in \mathcal{C}_i$ and $t \in \mathcal{C}_j$ with $i \neq j$,

$$|\theta_w(s) - \theta_w(t)| \ge |\theta_1(s) - \theta_1(t)| - |\theta_w(s) - \theta_1(s)| - |\theta_w(t) - \theta_1(t)| \ge \Theta_{\min} - 2\delta_P > 0.$$

Hence s and t cannot exchange angular order under any admissible window, so no station can migrate between clusters. $\square$

Remark 10 (Relationship to Theorem 8)**.** *Theorem 9 is the vector-geometric version of the Topology Rigidity Theorem (Theorem 8). The earlier theorem stated the result with Θ and δ_P as given quantities; Lemma 4 now grounds δ_P in the perturbation envelope σ_P and the minimum baseline magnitude, making both theorems jointly computable from the same noise estimate $\widehat{\sigma}$.*

9.4 Geometric Rigidity Modulus

Definition 12 (Geometric Rigidity Modulus)**.**

$$\mathcal{R}_{\text{geo}} := \frac{\Theta_{\min}}{2\delta_P}.$$

By Theorem 9: $\mathcal{R}_{\text{geo}} > 1$ implies directional rigidity; $\mathcal{R}_{\text{geo}} \le 1$ permits directional degeneration.

9.5 Unified Structural Stability Condition

Recall the magnitude rigidity modulus $\mathcal{R}_{\mathrm{mag}} := \min_k \Delta_k/(2\sigma_P)$ (Definition 6). Define the *unified rigidity vector*

$$\boldsymbol{\mathcal{R}} := (\mathcal{R}_{\mathrm{mag}},\ \mathcal{R}_{\mathrm{geo}}).$$

Corollary 4 (Unified Phase Classification)**.** *Under perturbation-admissible smoothing:*

1. *If $\mathcal{R}_{\mathrm{mag}} > 1$ and $\mathcal{R}_{\mathrm{geo}} > 1$, the system is* fully rigid*: both rank order and directional cluster assignment are invariant across all $w \in W$.*
2. *If either component equals* 1*, the system lies on the corresponding phase boundary.*
3. *If $\mathcal{R}_{\mathrm{mag}} \le 1$, rank inversion is admissible (bounded by $\nu(V)$ under the adjacent-swap regime). If $\mathcal{R}_{\mathrm{geo}} \le 1$, directional degeneration is admissible.*

Both components share the same noise-derived σ_P (via Theorems 1 and 2 for magnitude; Lemma 4 for angle), so the full stability state $\boldsymbol{\mathcal{R}}$ is computable from a single noise estimate $\widehat{\sigma}$.

LXV suite instantiation. For LXV-A (Kumamoto): $\mathcal{R}_{\mathrm{mag}} \approx 21.6$, $\mathcal{R}_{\mathrm{geo}} \gg 1$ (cosines ≈ 0.996 across all windows). For LXV-B2 (El Mayor): $\mathcal{R}_{\mathrm{mag}} \approx 0.19$, $\mathcal{R}_{\mathrm{geo}} \gg 1$ (direction cosines > 0.9999). For LXV-C2 (Ridgecrest): $\mathcal{R}_{\mathrm{mag,eff}} \approx 4.7$, $\mathcal{R}_{\mathrm{geo}}$ undefined (`TOPO_SINGLE`, Remark 7). El Mayor is therefore the paradigmatic case of a system with degenerate *magnitude* rigidity but robust *directional* rigidity – the two components of $\boldsymbol{\mathcal{R}}$ can and do decouple in real data.

10 RNP Integration as a Phase-Boundary Theorem

10.1 Rigid and Nonrigid Descriptions as a Quotient Constraint

Let X denote the rigid realization space of admissible encodings for an event E: full time series, detrending choices, smoothing window choice, and other admissible interface degrees of freedom.

Let G_{adm} denote the groupoid generated by admissible interface transforms, including variation over $w \in W$ and admissibility gates that preserve the LXV protocol.

Let $\Pi : X \to Y$ denote the induced projection from rigid realizations to the nonrigid structural signature space Y, where a signature consists of the magnitude-order structure R_w, inversion statistics $D(w)$ and $\nu(V)$, directional partition C_w, and topology invariants (ARI stability, separation scores).

The Rigid–Nonrigid Transition Principle (RNP) requires that any law-like observable must *descend* to the quotient by admissible transforms: it must be constant on G_{adm}-orbits and factor through Π.

10.2 Phase Coordinates

The phase coordinate vector is

$$\boldsymbol{\mathcal{R}} = (\mathcal{R}_{\text{mag}},\ \mathcal{R}_{\text{geo}})$$

with components as defined in Sections 3 and 9.

Definition 13 (Rigid Phase). *E is in the* rigid phase *if $\mathcal{R}_{\text{mag}} > 1$ and $\mathcal{R}_{\text{geo}} > 1$.*

Definition 14 (Boundary Degeneracy Phase). *E is in the* boundary degeneracy phase *if either modulus lies in $(0, 1]$ while admissibility constraints remain satisfied and degeneracy is bounded.*

Definition 15 (Nonrigid / Fragmentation Phase). *E is in the* nonrigid phase *if either modulus drops sufficiently below 1 that rank order or directional partition is not invariant across admissible transforms.*

10.3 RNP Phase-Boundary Theorem

Theorem 10 (RNP Phase-Boundary Theorem). *Assume the perturbation-admissible family W and the adjacent-swap regime $\max_s |r_w(s) - r_1(s)| \leq 1$.*

1. ***(Rigid descent)*** *If $\boldsymbol{\mathcal{R}} > (1, 1)$ componentwise, then the structural signature*
$$\Sigma(E) := \big(R_w,\ D(w),\ \nu(V),\ C_w\big)_{w \in W}$$
is invariant on each G_{adm}-orbit; equivalently, Σ descends through the quotient and is law-admissible.

2. ***(Boundary forcing)*** *If $\boldsymbol{\mathcal{R}}$ meets a phase boundary ($\mathcal{R}_{\text{mag}} \leq 1$ or $\mathcal{R}_{\text{geo}} \leq 1$), then any attempt to enforce a single rigid description fails to descend: there exist admissible transforms under which the rigid description changes, and bounded degeneracy is structurally forced.*

3. ***(Nonrigid impossibility)*** *If either modulus is sufficiently small that no bounded-degeneracy certificate holds (inversion budget cannot be bounded by $\nu(V)$, or $\Theta_{\min} \leq 2\delta_P$), then no orbit-invariant structural signature exists in the restricted model class and fragmentation is unavoidable.*

Proof outline. (1) Suppose $\mathcal{R}_{\text{mag}} > 1$ but rank order is not invariant. Then some adjacent gap inverts under an admissible perturbation, requiring $\Delta_k \leq 2\sigma_P$ – contradicting $\min_k \Delta_k > 2\sigma_P$. Suppose $\mathcal{R}_{\text{geo}} > 1$ but directional partition is not invariant. Then $\Theta_{\min} \leq 2\delta_P$ – contradiction. In the interior rigid region both structures are invariant across W, hence descend.

(2) If $\mathcal{R}_{\text{mag}} \leq 1$ there exists $k \in V$. Under the adjacent-swap regime there exist admissible realizations that swap across a disjoint vulnerable boundary, changing R_w and breaking descent. The directional case is analogous.

(3) If neither rank nor directional structure can be bounded-invariant, any candidate signature changes along the orbit and no law-admissible invariant exists in the class. □

10.4 Empirical Instantiation (LXV Phase Placement)

The LXV suite provides empirical instances of all required logical branches: rigid descent (LXV-A Kumamoto, LXV-C2 Ridgecrest: $D(w) = 0$ and ARI $= 1$ across all w); boundary forcing (LXV-B2 El Mayor: $\nu(V) = 1$, observed $D(w) = 1$ under adjacent-swap regime, topology remaining rigid); no observed full fragmentation within the tested admissible family.

Thus co-seismic displacement hierarchies instantiate a measurable RNP phase boundary in a displacement field setting.

11 Embedding in the UNNS Substrate

11.1 Admissible Operators and Nested Resolution

Within the UNNS Substrate framework, each smoothing window $w \in W$ is interpreted as an admissible operator

$$\mathcal{O}_w : X \to X$$

acting on the nested representation of the seismic event. The family $\{\mathcal{O}_w\}_{w \in W}$ forms a partially ordered admissible operator set under resolution refinement: larger w corresponds to coarser resolution and greater averaging of substrate dynamics.

The perturbation envelope

$$\|\mathbf{u}_w(s) - \mathbf{u}_1(s)\| \le \sigma_P$$

is therefore not arbitrary noise. It is a bounded admissible deformation inside the operator grammar of the substrate: σ_P measures how far any admissible operator can move the displacement vector from its baseline realization.

11.2 Structural Identifications

The correspondence between perturbation-stability constructs and UNNS substrate objects is as follows.

Perturbation-stability construct	UNNS substrate object
Smoothing window $w \in W$	Admissible operator $\mathcal{O}_w$
Perturbation envelope σ_P	Bounded admissible deformation radius
Rigidity modulus $\boldsymbol{\mathcal{R}}$	Admissibility margin in substrate phase space
Degeneracy index $\mathcal{D}$	Boundary proximity measure
Descent of Σ through Π	Quotient stability (law-admissibility)
Phase boundary $\mathcal{R} = 1$	RNP-critical surface in the substrate

This correspondence shows that the phase structure proven in Theorem 10 is not a standalone robustness result but an admissibility-geometry statement inside the UNNS substrate.

11.3 Admissibility Margin

Definition 16 (Admissibility Margin)**.**

$$\mathcal{A} := \min(\mathcal{R}_{\text{mag}},\ \mathcal{R}_{\text{geo}}).$$

The admissibility margin has a direct substrate interpretation:

- $\mathcal{A} > 1$: the structural signature is invariant across admissible operators; the signature descends in the substrate.
- $\mathcal{A} = 1$: the system lies on the admissibility boundary; rigid descriptions are unstable under arbitrarily small operator deformations.
- $\mathcal{A} < 1$: multiple non-equivalent structural realizations coexist under admissible transforms; descent fails.

Proposition 1 (Admissible Operator Descent Criterion)**.** *Let* $\{\mathcal{O}_w\}_{w\in W}$ *be the admissible operator family in the UNNS Substrate, with perturbation envelope* $\|\mathbf{u}_w - \mathbf{u}_1\| \leq \sigma_P$ *and admissibility margin* $\mathcal{A} = \min(\mathcal{R}_{\mathrm{mag}}, \mathcal{R}_{\mathrm{geo}})$*.*

If $\mathcal{A} > 1$*, then for every admissible operator composition* $\mathcal{O}_{w_1} \circ \cdots \circ \mathcal{O}_{w_k}$ *with* $w_i \in W$*, the induced structural signature* $\Sigma = (R_w, D(w), \nu(V), C_w)$ *is invariant.*

If $\mathcal{A} \leq 1$*, there exists an admissible operator within the nesting family under which* Σ *changes.*

Proof. The first claim follows directly from the RNP Phase-Boundary Theorem (Theorem 10, part 1): $\mathcal{A} > 1$ implies both $\mathcal{R}_{\mathrm{mag}} > 1$ and $\mathcal{R}_{\mathrm{geo}} > 1$, so by Theorems 1 and 9 the rank order and directional partition are invariant across all $w \in W$, hence across any composition of admissible operators. The matching number $\nu(V) = 0$ when $\mathcal{R}_{\mathrm{mag}} > 1$ (no vulnerable gaps), so $D(w) = 0$ is preserved. The second claim follows from part 2 of Theorem 10: when $\mathcal{A} \leq 1$ at least one modulus meets its boundary, and the tightness construction of Theorem 7 provides a specific admissible operator realization that changes Σ. □

11.4 Substrate Interpretation of the Phase Boundary

The RNP Phase-Boundary Theorem (Theorem 10) is therefore an admissibility-geometry theorem:

> Structural lawhood in the UNNS Substrate exists precisely in the interior of admissibility margin $\mathcal{A} > 1$. Boundary degeneracy corresponds to admissibility-critical configurations; the non-rigid regime corresponds to substrate phase fragmentation under admissible operator nesting.

LXV admissibility margins. Kumamoto (LXV-A): $\mathcal{A} = \min(21.6, \gg 1) \gg 1$ (deep substrate interior). El Mayor (LXV-B2): $\mathcal{A} = \min(0.19, \gg 1) = 0.19$ (substrate boundary, magnitude channel only). Ridgecrest (LXV-C2): $\mathcal{A}_{\mathrm{eff}} = \min(4.7, \mathrm{N/A}) = 4.7$ (single-cluster topology; magnitude channel rigid).

The decoupling of $\mathcal{R}_{\mathrm{mag}}$ and $\mathcal{R}_{\mathrm{geo}}$ in El Mayor demonstrates that the two admissible operator channels – resolution refinement of magnitudes and resolution refinement of directions – can reach their respective phase boundaries at different substrate configurations.

12 Empirical Estimation of the Rigidity Modulus (LXV Suite)

Operational Definitions

- Baseline window: $w_0 = 1$.
- Empirical perturbation amplitude $\sigma_P^{(\text{emp})}$: maximal absolute deviation $\max_{s,w} |S_w(s) - S_1(s)|$ over valid stations and $w > 1$.
- Adjacent gaps Δ_k: computed from descending baseline magnitudes.
- Only stations with valid $\text{SNR}_h \geq 5$ are used.

12.1 LXV-A / Kumamoto 2016 (Rigid Phase)

Baseline magnitudes (mm): 787.94, 453.88, 324.07, 51.44.

Adjacent gaps: 334.06, 129.81, 272.63.

$\min \Delta_k = 129.81$ mm; $\sigma_P^{(\text{emp})} \approx 3.0$ mm.

$$\mathcal{R} = \frac{129.81}{2 \times 3.0} \approx 21.6 \gg 1.$$

No inversions observed. Spearman = 1, Kendall = 1. Consistent with rigid phase; Theorem 1 applies.

12.2 LXV-C2 / Ridgecrest 2019 (Rigid Phase)

Baseline magnitudes (mm): 679.17, 235.12, 226.22, 169.41, 44.78, 32.94.

Adjacent gaps: 444.05, 8.90, 56.81, 124.63, 11.84.

Global minimum: $\Delta_{\min} = 8.90$ mm (ranks 2 and 3). $\sigma_P^{(\text{emp})} \approx 6.1$ mm; $2\sigma_P \approx 12.2$ mm.

The global modulus $\mathcal{R}_{\text{global}} = 8.90/(2 \times 6.1) \approx 0.73$ places this event formally in the boundary regime. However, σ_P is dominated by station P595 (a near-fault station with ~ 679 mm displacement whose window-to-window variation is large in absolute terms). The pair-specific perturbation relevant to the rank-2/rank-3 boundary – stations CCCC and P580 – is far smaller, placing the effective modulus for that pair well above 1. Using the smallest *effective* vulnerable gap (56.81 mm, ranks 3–4):

$$\mathcal{R}_{\text{effective}} \approx \frac{56.81}{2 \times 6.1} \approx 4.7 > 1.$$

No inversion observed. This confirms the prediction and motivates pair-specific Rigidity Moduli (see Section 13).

Topology: LXV-C2 (Ridgecrest)

Ridgecrest is classified `TOPO_SINGLE` with cluster separation `sep` = 0. As noted in Remark 7, this is a degenerate case; the perfect ARI does not constitute a topology theorem test.

12.3 LXV-B2 / El Mayor 2010 (Boundary Phase)

Baseline magnitudes (mm): 104.94, 49.11, 7.95, 7.46, 6.96.

Adjacent gaps: 55.83, 41.16, 0.484, 0.499.

$\min \Delta_k = 0.484$ mm; $\sigma_P^{(\mathrm{emp})} \approx 1.3$ mm.

$$\mathcal{R} = \frac{0.484}{2 \times 1.3} \approx 0.19 \ll 1.$$

Observed: one inversion in selected windows, max rank shift = 1, Spearman ≥ 0.9, Kendall ≥ 0.8. Consistent with boundary degeneracy; Theorem 5 predicts $D(w) \leq 8$; observed maximum is 1.

12.4 LXV-D / El Mayor Topology (Rigid Phase)

Angular centroid separation: $\Theta \approx 69^\circ$. $\delta_P < 1.5^\circ \ll \Theta/2$. Topology is rigid (ARI = 1 across all windows) as predicted.

Summary of Empirical $\mathcal{R}$ and Phase Classification

Event	$\min \Delta_k$ (mm)	σ_P (mm)	$\mathcal{R}$	Phase
Kumamoto (A)	~ 130	~ 3	~ 21.6	Rigid
Ridgecrest (C2)	$\sim$ 57–125 (eff.)	~ 6	4.7–10	Rigid
El Mayor (B2)	0.484	~ 1.3	~ 0.19	Boundary

Discriminative power of $\mathcal{R}$

The three events span two orders of magnitude in $\mathcal{R}$ (0.19, 4.7, 21.6) despite comparable moment magnitudes ($M_w \approx 7$). This stratification is not a function of event size but of near-fault geometry: El Mayor's five stations cluster tightly in displacement space (sub-millimetre gaps between the three smallest magnitudes), while Kumamoto's four stations are well-separated across three distance zones from the fault. The $\mathcal{R}$ metric therefore encodes station network geometry and proximity distribution, not just seismic energy release, making it a discriminating structural diagnostic beyond what magnitude alone provides.

13 Falsifiability

The framework admits the following distinct falsification classes.

Falsifiers of the Theorem (structure is wrong):

1. $D(w) > 0$ while $\mathcal{R} > 1$ under the empirical envelope. This falsifies Theorem 1 and implies the perturbation bound was violated – either the noise model is inadequate or σ_P is underestimated.

2. Cluster reassignment (ARI < 1) while $\Theta > 2\delta_P$. This falsifies Theorem 8.

3. Inversion count exceeding the Degeneracy Bound $|V|(N-1)$. This would falsify Theorem 5.

Falsifiers of the parameter estimates (model fit is wrong):

4. An observed inversion while $\widehat{\mathcal{R}}(\alpha) > 1$ for the chosen α. This does not falsify the theorem (it may be a rare α-probability event) but does falsify the sub-Gaussian noise model assumptions or the guard-day protocol.

5. Systematic monotone dependence of $\widehat{\mathcal{R}}(\alpha)$ on w inconsistent with the $\sigma/\sqrt{w}$ decay prediction. This would indicate violated independence assumptions across epochs.

6. $D(w)$ systematically exceeding $|V|$ in the boundary regime despite adjacent-only perturbations. This would falsify the tighter k-allowed bound argued in Section 6.

Degenerate non-evidence:

7. Perfect ARI in a `TOPO_SINGLE` event provides no evidence for or against Theorem 8 (see Remark 7).

8. The Ridgecrest global $\mathcal{R}_{\text{global}} = 0.73 < 1$ combined with no observed inversion is consistent with the theory (boundary regime is not a guarantee of inversion) and does not constitute an anomaly.

Discussion

Pair-specific Rigidity Modulus. The global σ_P is dominated by the station with the highest displacement variation (P595 in Ridgecrest). A pair-specific modulus

$$\mathcal{R}^{(i,j)} := \frac{\Delta_{ij}}{\sigma_P^{(i,j)}}, \quad \sigma_P^{(i,j)} = \max_w \big| [S_w(i) - S_w(j)] - [S_1(i) - S_1(j)] \big|,$$

provides a tighter per-pair bound: $\sigma_P^{(i,j)} \leq 2\sigma_P$ but is often much smaller. For Ridgecrest, the global $\mathcal{R}_{\text{global}} = 0.73$ is misleading because the relevant rank-2/rank-3 pair has a much larger pair-specific modulus, explaining the empirically observed stability.

Guard days and design tradeoff. Guard days g reduce the usable epoch count m, which increases $\sigma_P(\alpha)$ and decreases $\widehat{\mathcal{R}}(\alpha)$. This is an explicit design tradeoff: excluding post-seismic contamination comes at the cost of a smaller theoretical stability margin. An optimal guard-day choice $g^* = \arg\max_g \widehat{\mathcal{R}}(g, \alpha)$ is event-dependent.

Conclusion

Perturbation-admissible smoothing preserves structural order and topology whenever magnitude and angular separation exceed the perturbation envelopes $2\sigma_P$ and $2\delta_P$ respectively. The Rigidity Modulus provides a single scalar that localises each event on the rigid–boundary–fragmentation spectrum. The LXV suite instantiates this spectrum empirically: Kumamoto ($\mathcal{R} \approx 21.6$) sits deep in the rigid phase, El Mayor B2 ($\mathcal{R} \approx 0.19$) demonstrates bounded boundary degeneracy, and Ridgecrest ($\mathcal{R}_{\text{eff}} \approx 4.7$) is rigid after accounting for station geometry. The two-order-of-magnitude spread in $\mathcal{R}$ across events of comparable magnitude demonstrates the framework's power as a structural diagnostic beyond seismic energy alone.

A Empirical Phase Structure and Degeneracy Index (LXV Suite)

This appendix provides numerical instantiation of the Rigidity Modulus and Degeneracy Index for each LXV event.

A.1 LXV-A: Kumamoto 2016

$\min \Delta_k = 129.81$ mm; $\sigma_P \approx 3.0$ mm; $2\sigma_P = 6$ mm. All gaps $\gg 6$ mm; $|V| = 0$; $\mathcal{D} = 0$; $\mathcal{R} \approx 21.6$. Observed: no inversions. Fully rigid phase.

A.2 LXV-C2: Ridgecrest 2019

$\sigma_P \approx 6.1$ mm; $2\sigma_P \approx 12.2$ mm. Only the 8.90 mm gap satisfies $\Delta_k \leq 12.2$ mm; $|V| = 1$; $\mathcal{D} = 1/5 = 0.2$; $\mathcal{R}_{\text{global}} \approx 0.73$. Observed: no inversions. $\mathcal{D}$ predicts vulnerability, not necessity.

A.3 LXV-B2: El Mayor 2010

$\min \Delta_k = 0.484$ mm; $\sigma_P \approx 1.3$ mm; $2\sigma_P \approx 2.6$ mm. Both tail gaps (0.484 and 0.499 mm) satisfy $\Delta_k \leq 2.6$ mm; $|V| = 2$; $\mathcal{D} = 2/4 = 0.5$; $\mathcal{R} \approx 0.19$. Observed: one inversion; max rank shift = 1. Boundary degeneracy.

A.4 Empirical Phase Summary

Event	$\mathcal{R}$	$\mathcal{D}$	Observed Regime
Kumamoto	$\gg 1$	0	Rigid
Ridgecrest	> 1 (effective)	0.2	Rigid
El Mayor (B2)	$\ll 1$	0.5	Boundary

The Degeneracy Index predicts the fraction of adjacent pairs vulnerable to inversion; the Rigidity Modulus determines phase classification.