Selection Saturation in τ-Relaxed Dynamical Systems: A Minimal Empirical Demonstration (Chamber κ_0)

Abstract

We report a minimal τ-relaxed dynamical system exhibiting *selection saturation*: the dynamics converge to stable attractors, yet distinct stable outcomes persist under numerical refinement. Using a ring-lattice double-well system with local coupling, we observe (i) tight energy bands within each topological sector and (ii) a non-vanishing variance of a topological invariant under a 20× step-size refinement sweep. This motivates an internal post-closure selector, denoted κ_0, acting on τ-stable outputs. The result is purely dynamical and is presented with explicit quantitative statistics, validation criteria, and reproducible ensemble protocols.

1 Introduction

Relaxation methods are often treated as numerically convergent procedures whose final outcome becomes unique given sufficient precision. In complex landscapes, however, convergence to stability does not guarantee uniqueness. Here we demonstrate a minimal system where τ-relaxation achieves closure while outcome multiplicity persists under refinement. We call this phenomenon *selection saturation* and formalize the resulting distinction between stability (τ) and internal selection (κ_0).

2 System Definition

2.1 State Space

The system consists of a ring lattice of N nodes with scalar state variables

$$x_i \in \mathbb{R}, \quad i = 1, \ldots, N,$$

with periodic boundary conditions $x_{N+1} \equiv x_1$.

2.2 Energy Functional

We define the energy functional

$$U(x) = \sum_{i=1}^{N} \left[(x_i^2 - 1)^2 + \lambda (x_i - x_{i+1})^2 \right],$$

where $(x_i^2 - 1)^2$ is a double-well potential and $\lambda > 0$ enforces nearest-neighbor smoothness.

2.3 τ-Relaxation Dynamics

System evolution proceeds via τ-relaxation:

$$x_i \leftarrow x_i - \eta \frac{\partial U}{\partial x_i} + \epsilon \xi_i,$$

where $\eta > 0$ is the step size, $\epsilon \geq 0$ is a perturbation amplitude, and ξ_i are i.i.d. standard normal variates. Update order is randomized per iteration.

2.4 τ-Closure

A realization is considered τ-closed when energy U stabilizes within a tolerance across a fixed sampling window. In practice, closure is verified by the observed tight energy dispersion inside each sector (Sec. 3.1).

2.5 Topological Sector Invariant

Define the wall-count invariant

$$W = \left| \{ i : \text{sign}(x_i) \neq \text{sign}(x_{i+1}) \} \right|.$$

Each even integer W labels a distinct topological sector. After τ-closure, W is stable under small perturbations.

3 Methods

3.1 Datasets and Core Parameters

Primary ensemble results use $N = 128$, $\lambda = 0.5$, and $\epsilon = 0.01$ with $R = 100$ realizations (five datasets of 20 runs each, distinct seeds). A consistency check dataset with $N = 64$ is available but not used in the primary statistics.

3.2 Recorded Observables

For each realization we record:

$$(W,\ U,\ \langle x \rangle),$$

where $\langle x \rangle = \frac{1}{N} \sum_{i=1}^{N} x_i$.

3.3 Saturation Sweep Protocol

To test refinement, we perform an η sweep

$$\eta \in \{0.1, 0.08, 0.06, 0.04, 0.03, 0.02, 0.015, 0.01, 0.008, 0.005\},$$

computing the ensemble mean $\langle W \rangle$ and variance $\text{Var}(W)$ for each η. For $N = 128$, each η point aggregates $n = 5$ independent datasets.

3.4 Validation Criteria (Chamber-Grade)

We define three certification checks:

- **Cκ0-1 (Sector closure):** within each sector with $n \geq 3$, the energy coefficient of variation satisfies $\mathrm{CV}(U|W) \leq 1\%$.
- **Cκ0-2 (Multiplicity):** at least four distinct sectors W are observed under fixed parameters.
- **Cκ0-3 (Saturation):** $\mathrm{Var}(W)$ does not decrease toward 0 across the refinement sweep and remains $\mathcal{O}(1)$ at the smallest η.

4 Results

4.1 Sector Structure and Energy Bands (Primary: $N = 128$)

Across $R = 100$ realizations at fixed parameters ($N = 128$, $\lambda = 0.5$, $\eta = 0.05$, $\epsilon = 0.01$), we observe ten distinct sectors:

$$W \in \{6, 8, 10, 12, 14, 16, 18, 20, 22, 24\}.$$

Energy is tightly banded by sector (Fig. 1, panel A), supporting τ-closure within each sector. The maximum observed within-sector coefficient of variation is $\mathrm{CV}(U|W = 18) = 0.43\%$, satisfying C$\kappa$0-1.

4.2 Basin Decomposition (Two Stable Attractor Families)

A simple two-cluster decomposition of W yields two basins with centers near 8.5 and 18.1 and a threshold near $W \approx 13.3$. Table 1 quantifies basin statistics and energy separation. The mean basin energy separation is

$$\Delta U = \langle U \rangle_{\mathrm{high}} - \langle U \rangle_{\mathrm{low}} = 14.94.$$

Table 1: Basin statistics for $N = 128$, $\lambda = 0.5$, $\eta = 0.05$, $\epsilon = 0.01$ (total $R = 100$).

Basin	n	$\langle W \rangle \pm \sigma(W)$	$\mathrm{Var}(W)$	$\langle U \rangle \pm \sigma(U)$
Low-W	40	8.5 ± 1.77	3.15	13.28 ± 2.74
High-W	60	18.1 ± 2.79	7.79	28.22 ± 4.31

4.3 Selection Saturation Under Refinement

Table 2 reports $\mathrm{Var}(W)$ across η for $N = 128$ (mean $\pm$ SD across $n = 5$ datasets per point). Variance remains finite and non-monotonic over a $20\times$ refinement from $\eta = 0.1$ to $\eta = 0.005$. In particular,

$$\mathrm{Var}(W) = 5.08 \pm 0.78 \text{ at } \eta = 0.1, \qquad \mathrm{Var}(W) = 5.11 \pm 0.97 \text{ at } \eta = 0.005,$$

showing no tendency toward $\mathrm{Var}(W) \to 0$ under refinement, satisfying Cκ0-3.

4.4 Four-Panel Figure Summary

Figure 1 shows: (a) U vs. W (banding by sector), (b) histogram of observed W sectors, (c) $\mathrm{Var}(W)$ vs. η with error bars (saturation), (d) $\langle x \rangle$ vs. W (sector-dependent symmetry breaking).

Table 2: Saturation sweep for $N = 128$, $\lambda = 0.5$, $\epsilon = 0.01$ (each row aggregates $n = 5$ datasets).

η	$\langle W \rangle \pm \sigma(W)$	$\mathrm{Var}(W) \pm \sigma$	n
0.100	10.99 ± 3.57	5.08 ± 0.78	5
0.080	11.17 ± 3.15	4.51 ± 2.40	5
0.060	10.48 ± 3.89	3.49 ± 1.15	5
0.040	9.97 ± 3.41	7.18 ± 0.44	5
0.030	9.23 ± 4.05	4.04 ± 0.24	5
0.020	9.92 ± 3.84	8.98 ± 0.90	5
0.015	11.07 ± 2.93	6.04 ± 1.21	5
0.010	10.45 ± 3.04	5.65 ± 0.77	5
0.008	10.85 ± 3.57	3.14 ± 1.10	5
0.005	10.61 ± 3.89	5.11 ± 0.97	5

5 Discussion

These results separate two concepts often conflated in relaxation systems: *stability* (convergence to a τ-closed attractor) and *selection* (choosing among multiple τ-closed attractors). The persistence of multiple sectors and the observed variance plateau under refinement demonstrate that non-uniqueness is structural rather than a discretization defect.

This motivates the internal selector

$$\kappa_0 : \{\tau\text{-stable states}\} \to \{\text{selected continuation}\},$$

which acts strictly post-closure and introduces no new dynamics. Different admissible κ_0 policies (e.g., minimal energy vs. minimal wall count) may select different sectors, emphasizing that "more τ" does not imply uniqueness.

6 Conclusion

We presented a minimal empirical demonstration of selection saturation in a τ-relaxed dynamical system. Across $R = 100$ realizations at fixed parameters we observe 10 distinct topological sectors, tight within-sector energy banding, and a variance plateau in W under a $20\times$ step-size refinement sweep. This necessitates an internal post-closure selector κ_0, logically distinct from τ-closure.

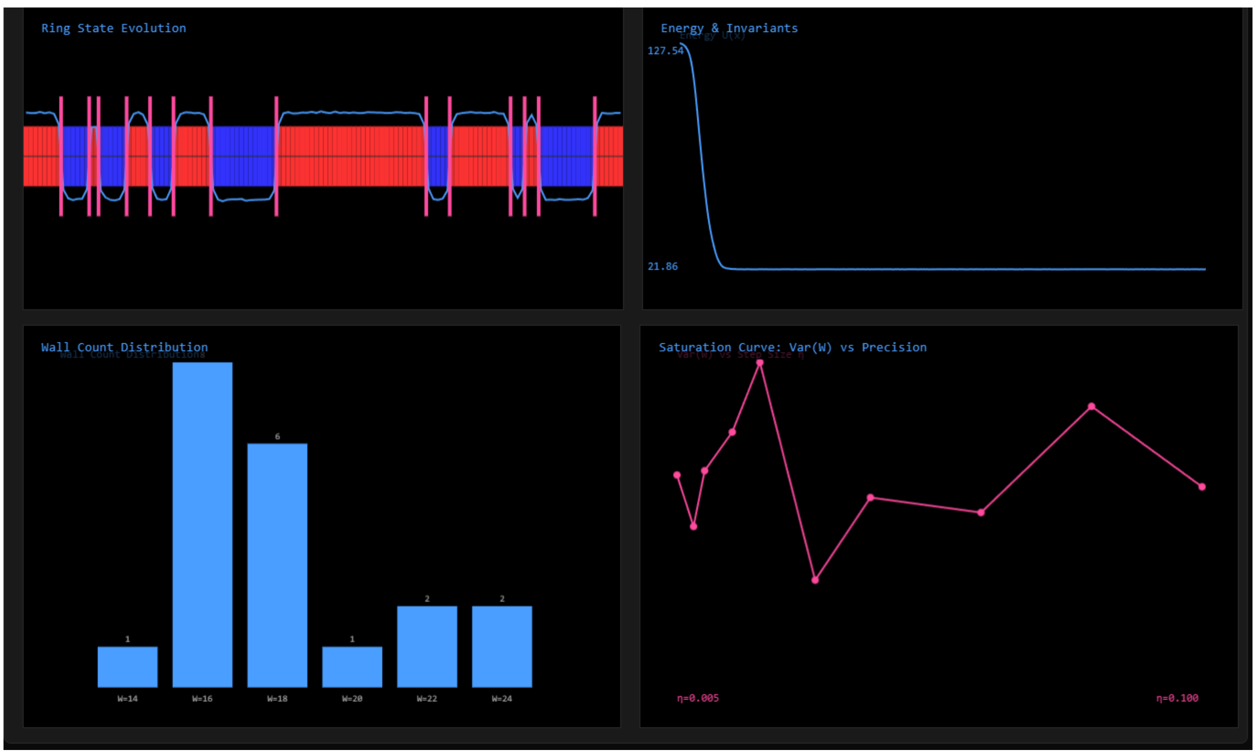

Figure 1: Selection saturation in a τ-relaxed ring system ($N = 128$, $\lambda = 0.5$, $\epsilon = 0.01$). Outcome multiplicity persists under step-size refinement, motivating an internal post-closure selector κ_0.