

THE MATHEMATICAL FOUNDATIONS
OF THE UNNS SUBSTRATE

Volume II

Collective Dynamics, Structural Rigidity,
and Effective Predictability

From Generative Multiplicity to Exact Scaling
and Arithmetic Certification

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Abstract

Volume I determined what the UNNS Substrate generates: which nested occurrences realise a given value, how their routes meet again, how rigidly they are distinguished, and why no interaction law is forced by the arithmetic alone. This volume asks what happens when generative structures evolve *collectively*. We fix one explicit interaction rule, the collective map $C(x) = x + L(x)^2/G(x)$ on the positive rationals, where L and G are the least common multiple and the greatest common divisor of the moduli carried by the occurrence fibre of x . Once the rule is selected, its dynamics are governed by theorems, and five of them organise the volume.

Unbounded generative multiplicity. Along every orbit the fibre size $R(x_n)$ is non-decreasing and tends to infinity, the gcd $G(x_n)$ can only decrease through divisors, and the reach set, although infinite, has logarithmic counting function and density zero. *Eventual exact scaling.* Every orbit satisfies $x_{n+P} = \rho^2 x_n$ from some time on. The proof passes through an arithmetic state description in which a persistent orbit is a finite-state process driven by a single integer coordinate γ and residue classes of prime exponents; the persistent regimes with $G_\infty > 1$ exist for every $g \geq 5$ prime to 6 and for several g divisible by 3, G_∞ never divides 24, and the eight-step orbit at $g = 11$ shows that no universal three-step law holds. *Universal single-prime rigidity.* The carries of a persistent orbit form a system of independent cyclic counters read by a scheduler that always fires the smallest pending carry; exactly one prime carries infinitely often, and the scaling factor is a prime power $\rho = p^s$. *Arithmetic survivor selection.* In a system of independent odometers with fixed tables, the survivor is the prime whose worst-case carry is smallest; a quotient-residue formula computes that carry from finite residue data; and, at a reset state of an orbit with prime g that satisfies the stated certificate hypotheses, a finite residue test decides the survivor and fixes the entire future. For an arbitrary initial state the survivor is determined by running the exact carry-level predictor to a reset state, within the transient bounds below. *Effective finite predictability.* From the initial pair (g_0, γ_0) alone, an explicit and computable bound limits the number of carry events before the orbit enters its final cycle; the reset-descent part of this bound, $\kappa(\gamma_0)$, is attained at each level $\kappa = 1, \dots, 11$ by explicit witnesses.

The volume develops all seven stages with complete proofs, states every hypothesis and scope, and supports each statement with exact integer computation reproducible from the accompanying package.

Contents

Abstract	i
Introduction: from generative structure to predictive dynamics	1
1 Collective Grammar and the Emergence of Structural Growth	3
1.1 The collective map and its arithmetic frame	3
1.2 Plateaus and the time to the next growth	4
1.3 Unbounded generative multiplicity	5
1.4 Growth is specific to the collective branch	6

2	Arithmetic Regimes and the Evolution of Structural Invariants	6
2.1	The square-root part and the arithmetic of G	6
2.2	Regime I: closed dynamics at $G = 1$	7
2.3	Modular escape and the first exclusions	8
2.4	Transitions between fibres	9
2.5	The reach set and non-universality	10
3	Persistence, Admissibility, and Structural Exclusion	11
3.1	The γ -coordinate	11
3.2	Persistent regimes	12
3.3	Universal collapse is false	14
3.4	Exclusion of $G_\infty \mid 24$	14
3.5	A bounded window for γ	15
4	The Gamma Framework and Exact Asymptotic Scaling	16
4.1	The one-step law	16
4.2	The eight-step orbit and the failure of a uniform period	16
4.3	Finite-state closure and exact scaling	17
4.4	Which carries occur	19
4.5	The q -carry family	21
5	Carry Scheduling and Universal Single-Prime Rigidity	23
5.1	The carry scheduler	23
5.2	Single-prime dominance	24
5.3	Consequences	25
6	Predictive Selection and Arithmetic Certification	27
6.1	The exact carry-level predictor	27
6.2	The minimax selection theorem	28
6.3	The quotient-residue formula	28
6.4	The static certificate	29
6.5	Consequences of certification	30
7	Effective Transient Bounds and Finite Predictability	31
7.1	The reset-descent bound	32
7.2	The class-counting entry bound	32
7.3	The polynomial entry bound	34
7.4	The dynamic- G effective bound	35
7.5	Registered scope of the programme	36
7.6	Finite predictability	36
A	Mathematical definitions and notation	37
A.1	Objects	37
A.2	Scope tags	38
A.3	Correspondence with Volume I and with the source records	38
B	Theorem registry and proof dependencies	38
C	Exact arithmetic algorithms and reproducibility	42
C.1	Algorithms	42
C.2	Verification	43
D	Proof-completeness audit of the volume's records	45
E	Mathematical contributions and relation to existing theory	47
F	Open problems and further mathematical development	48

Introduction: from generative structure to predictive dynamics

Where Volume I ends. Volume I established the arithmetic of the Substrate. For a positive rational $x = a/b$ in lowest terms, the occurrences carrying the value x are indexed by the *moduli*

$$\mathcal{D}(x) = \{d \geq 2 : d^2 \mid a, d \equiv 1 \pmod{b}\}, \quad R(x) = |\mathcal{D}(x)|,$$

(Volume I, Theorem 1.2). Any two routes through x meet again exactly on the progression $x + j\Delta_{\text{coll}}$, and for the whole fibre

$$\Delta_{\text{coll}}(x) = \frac{L(x)^2}{G(x)}, \quad L = \text{lcm } \mathcal{D}(x), \quad G = \text{gcd}\{d - 1 : d \in \mathcal{D}(x)\},$$

with the integral coherence invariant $\gamma(x) = x/\Delta_{\text{coll}}(x) \in \mathbb{Z}_{>0}$ (Volume I, Theorem 2.5). Volume I also proved that this arithmetic does not select a continuation rule: label-blind admissibility, stationarity and weak label-blindness cannot all hold, and any interaction must be specified. It ended by naming the rule studied here.

The collective map. The map

$$C(x) = x + \frac{L(x)^2}{G(x)} = x + \Delta_{\text{coll}}(x)$$

sends each value to the first value that *all* routes through it share in common, other than x itself. It is an explicitly selected interaction rule. Once it is selected, however, what its orbits do is not a further assumption: it is a matter of theorem. Whether the fibre grows, whether the gcd collapses, whether the orbit settles into a rhythm, which primes drive that rhythm, and how long the settling takes are questions with exact answers, and this volume proves them.

Five achievements. The proofs fall into five groups. Each states a different kind of knowledge: that something happens (existence), what constrains it (structure), and how soon (effectiveness).

Achievement	Statement	Where
Unbounded generative multiplicity	$R(x_n) \rightarrow \infty$ along every orbit; $D(x_n) \subseteq D(x_{n+1})$; the reach set is sparse	Ch. 1, Thm 1.8, 1.3; Cor. 5.7
Eventual exact scaling	$x_{n+P} = \rho^2 x_n$ eventually, for every orbit; explicit persistent families	Ch. 3–4, Thm 4.5, 4.3, 4.12
Universal single-prime rigidity	exactly one prime carries infinitely often; $\rho = p^s$	Ch. 5, Thm 5.2, 5.6
Arithmetic survivor selection	survivor = arg min of the worst-case carry; static certificate for prime g	Ch. 6, Thm 6.3–6.2
Effective finite predictability	carry events before the final cycle $\leq \mathcal{R}(g_0, \gamma_0)$; reset descent $\kappa(\gamma_0)$, attained at $\kappa \leq 11$	Ch. 7, Thm 7.2–7.6

Existence is the content of Chapters 1–4: eventual arithmetic behaviour is established for every orbit, with explicit regimes showing which behaviours occur. Structure is Chapters 5–6: the carries that drive the behaviour form a system of independent counters under a one-line scheduler, and that system admits exactly one survivor. Effectiveness is Chapter 7: finite

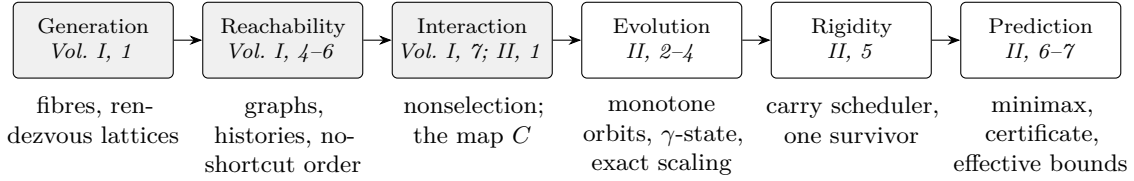


Figure 1: The two-volume progression. Shaded stages are the content of Volume I; Chapter 1 of this volume begins where the nonselection theorem of Volume I ends.

bounds, computable from the first arithmetic data (g_0, γ_0) , control the transient. The volume therefore goes beyond describing patterns or observing regularity numerically; each stage is a theorem with stated hypotheses.

Two precise scopes. Two qualifications govern how the predictive theorems are to be read, and they are stated here once and repeated in the statements of the theorems concerned.

(i) *Carries versus steps.* The principal bound of Theorem 7.6 counts *carry events*, the steps at which L changes, not iterations of C . Between carries the orbit runs through plateaus of at most $4g$ steps (Theorem 2.7), so the number of iterations is at most $(\text{carries} + 1) \cdot 4g + 1$ within each segment of constant G , together with the $G = 1$ tail estimate; the theorem records these additional estimates.

(ii) *Prime g for certification.* The static certificate (Theorem 6.5) and its consequence for the eventual regime (Theorem 6.8(a)) are proved for prime g , or more generally whenever no prime factor of g is at most m^2 . For composite and even g the survivor exists and the final cycle is entered within the class-counting bound, but static certification as formulated here is not available at the reset states of the six exact composite and even cycles examined (Theorem 7.7); this is open problem O12.2.

Reading the volume. Chapter 1 defines the dynamics and proves that the fibre grows without bound. Chapter 2 studies the invariant G , the regime $G = 1$ in closed form, and the transition mechanism between fibres. Chapter 3 shows that G need not collapse, constructs the persistent regimes, and introduces the coordinate γ . Chapter 4 proves the one-step law, the eight-step theorem, exact scaling for every orbit, and the carry structure. Chapter 5 derives the scheduler and single-prime dominance. Chapter 6 develops minimax selection and certification. Chapter 7 proves the effective bounds. The appendices give notation, the complete theorem crosswalk and audit, the algorithms and checks, the mathematical contributions, and the open problems. Every theorem carries a source-record tag (for example F.T51) referring to the FIELD-I records and the 128-record registry; Figure 1 shows how the two volumes fit together.

Conventions. All numbers are exact. Rationals are in lowest terms. v_p is the p -adic valuation, $\text{ord}_g(p)$ the multiplicative order of p modulo g , and lcm and gcd are as in Volume I. The orbit of x_0 is $x_{n+1} = C(x_n)$, and all quantities D_n, L_n, G_n, γ_n are those of x_n . A *scope tag* marks what a statement depends on: scope A0 depends on the nest rule alone, scope C on the adoption of C . Every result of this volume other than the two A0 records (F.T17p and F.T30) is scope C. Appendix A fixes the notation and its correspondence with Volume I and the FIELD-I symbols.

Chapter 1 Collective Grammar and the Emergence of Structural Growth

This chapter defines the dynamical system studied in the volume and proves its first laws: the orbit is well defined, the fibre only grows, the orbit moves along an integer progression between growth events, each growth event occurs within an explicit number of steps, and consequently the fibre grows without bound.

1.1 The collective map and its arithmetic frame

Throughout, a *realised* value is a positive rational $x = a/b$ in lowest terms with $\mathcal{D}(x) \neq \emptyset$, where $\mathcal{D}(x) = \{d \geq 2 : d^2 \mid a, d \equiv 1 \pmod{b}\}$. Write $R(x) = |\mathcal{D}(x)|$, $L(x) = \text{lcm } \mathcal{D}(x)$, $G(x) = \gcd\{d - 1 : d \in \mathcal{D}(x)\}$.

Definition 1.1 (The collective map). SOURCE RECORD: F.CDEF. Scope C. For a realised value x put

$$\Delta_{\text{coll}}(x) = \frac{L(x)^2}{G(x)}, \quad C(x) = x + \Delta_{\text{coll}}(x).$$

The *collective orbit* of a realised x_0 is $x_{n+1} = C(x_n)$. Write $\mathcal{D}_n, R_n, L_n, G_n$ for the quantities of x_n , and $\gamma(x) = x/\Delta_{\text{coll}}(x)$.

By Volume I (Theorem 2.5), $x + j\Delta_{\text{coll}}(x)$ ($j \geq 0$) are exactly the values shared by all routes through x at or after x . The map C selects $j = 1$: the first collective meeting. The definition is an adopted interaction rule; Volume I proved that the arithmetic leaves this choice open. The theorems below concern what the adopted rule does. The next proposition fixes the arithmetic frame in which every later statement is written; it uses nothing but the definition of $\mathcal{D}$.

Proposition 1.2 (The collective step is reduced and γ is an integer). SOURCE RECORD: F.T17P. Scope A0. Let $x = a/b$ be realised, $G = G(x)$, $L = L(x)$. Then

- (a) $b \mid G$;
- (b) $\gcd(L, G) = 1$; hence $\Delta_{\text{coll}} = L^2/G$ is in lowest terms and has denominator G ;
- (c) $L^2 \mid a$; the number $y = Gx = aG/b$ is an integer divisible by L^2 , and $\gamma(x) = y/L^2$ is a positive integer. Thus $x = \gamma L^2/G$.

Proof. (a) Each $d \in \mathcal{D}$ satisfies $b \mid d - 1$, so $b \mid G$. (b) A prime $p \mid G$ divides $d - 1$ for every $d \in \mathcal{D}$, hence divides no d , hence not L . (c) $d^2 \mid a$ for all $d \in \mathcal{D}$ gives $L^2 = \text{lcm}(d^2) \mid a$. By (a), G/b is an integer, so $y = a(G/b)$ is an integer divisible by L^2 . Finally $\gamma = x/\Delta_{\text{coll}} = Gx/L^2 = y/L^2$. $\square$

Theorem 1.3 (Monotone orbits). SOURCE RECORD: F.T31. Scope C. For every realised x : $\mathcal{D}(x) \subseteq \mathcal{D}(C(x))$ and $G(C(x)) \mid G(x)$. Consequently every collective orbit is defined for all n , $\mathcal{D}_n \subseteq \mathcal{D}_{n+1}$, and $G_{n+1} \mid G_n$.

Proof. Let $x = a/b$, $G = G(x)$, $L = L(x)$. Over the denominator G the value $x' = C(x)$ is N/G with $N = a(G/b) + L^2$. Take $d \in \mathcal{D}(x)$. Then $d^2 \mid a$, so $d^2 \mid a(G/b)$; also $d^2 \mid L^2$; hence $d^2 \mid N$. Since $G \mid d - 1$, we have $\gcd(d, G) = 1$, so d^2 divides the numerator of N/G in lowest terms, and $d \equiv 1$ modulo G , hence modulo the reduced denominator, a divisor of G . Thus

$d \in \mathcal{D}(x')$. The gcd defining $G(x')$ runs over a larger set, so $G(x') \mid G(x)$. Inductively every x_n is realised. $\square$

1.2 Plateaus and the time to the next growth

Between growth events the data $(\mathcal{D}, L, G)$ are constant and the orbit moves along an arithmetic progression.

Proposition 1.4 (Plateau law). SOURCE RECORD: F.T17. *Scope C. Suppose $\mathcal{D}_{n_0+k} = \mathcal{D}_{n_0}$ for $0 \leq k \leq K$. Then $x_{n_0+k} = (\gamma_{n_0} + k)L^2/G$ and $\gamma_{n_0+k} = \gamma_{n_0} + k$ for $0 \leq k \leq K$.*

Proof. L and G are constant on the plateau, so $x_{n_0+k} = x_{n_0} + k\Delta_{\text{coll}}$, and dividing by Δ_{coll} gives γ increasing by one per step. $\square$

Write $T(x)$ for the least $n \geq 1$ with $\mathcal{D}(C^n(x)) \neq \mathcal{D}(x)$, the time to the next growth. It is finite by the next theorem.

Theorem 1.5 (Growth-time bound). SOURCE RECORD: F.T18. *Scope C. Let x be realised, $\gamma = \gamma(x)$, $G = G(x)$, and let $d \geq 2$ be an integer with $d \notin \mathcal{D}(x)$. Put*

$$K_d = Gd^2 \left\lceil \frac{\gamma}{Gd^2} \right\rceil - \gamma.$$

Then $1 \leq K_d \leq Gd^2 - 1$ and $T(x) \leq K_d$. In particular $T(x) \leq Gd_{\min}^2 - 1$, where $d_{\min}$ is the least integer ≥ 2 not in $\mathcal{D}(x)$.

Proof. The ceiling gives $0 \leq K_d < Gd^2$. If $K_d = 0$, then $Gd^2 \mid \gamma$, so $x = \gamma L^2/G = (\gamma/G)L^2$ is an integer divisible by d^2 , whence $d \in \mathcal{D}(x)$ (a modulus condition modulo $b = 1$ is empty): a contradiction. So $K_d \geq 1$. Suppose no growth occurs in the steps $1, \dots, K_d$. By Proposition 1.4, $x_{K_d} = (\gamma + K_d)L^2/G = d^2 \lceil \gamma/(Gd^2) \rceil L^2$, an integer divisible by d^2 . Then $d \in \mathcal{D}(x_{K_d}) = \mathcal{D}(x)$, contradicting $d \notin \mathcal{D}(x)$. $\square$

The bound is an upper bound; actual growth is usually earlier because several moduli may become available at once. Its best form over all choices of d is computable by a finite search.

Proposition 1.6 (Exact minimum of the bound). SOURCE RECORD: F.T26. *Scope C. Let $k_0 = K_{d_{\min}}$. Then $\min\{K_d : d \geq 2, d \notin \mathcal{D}(x)\}$ is attained at some d with $d^2 < (\gamma + k_0)/G$ (or equals k_0); hence it is computed by a finite search, and $T(x)$ is at most this minimum.*

Proof. If $K_d < k_0$, then $Gd^2 \leq Gd^2 \lceil \gamma/(Gd^2) \rceil = \gamma + K_d < \gamma + k_0$, because $\gamma \geq 1$. Hence only d with $Gd^2 < \gamma + k_0$ can improve on k_0 . $\square$

Example 1.7. The orbit of $x_0 = 36$ has $\mathcal{D} = \{2, 3, 6\}$, $G = 1$, $L = 6$, $\Delta_{\text{coll}} = 36$ and runs 36, 72, 108 with $\gamma = 1, 2, 3$; at 144 the modulus 4 (and with it 12) appears. The orbit of the natural seed $x_0 = 45828/5$ has $\mathcal{D} = \{6\}$, $G = 5$, $\gamma_0 = 1273$, and acquires the moduli 21, 126, 36, ... within three steps:

n	0	1	2	3	4	5	6	7	9
$R(x_n)$	1	2	3	4	4	4	5	5	7
γ_n	1273	26	3	1	2	3	1	2	1
L_n	6	42	126	252	252	252	504	504	1008

Here $G_n = 5$ for every n , a persistent regime studied in Chapter 3; the large initial γ falls at the carries $6 \rightarrow 42 \rightarrow 126$ and the orbit settles into the cycle $\gamma = 1, 2, 3$ with a carry every third step.

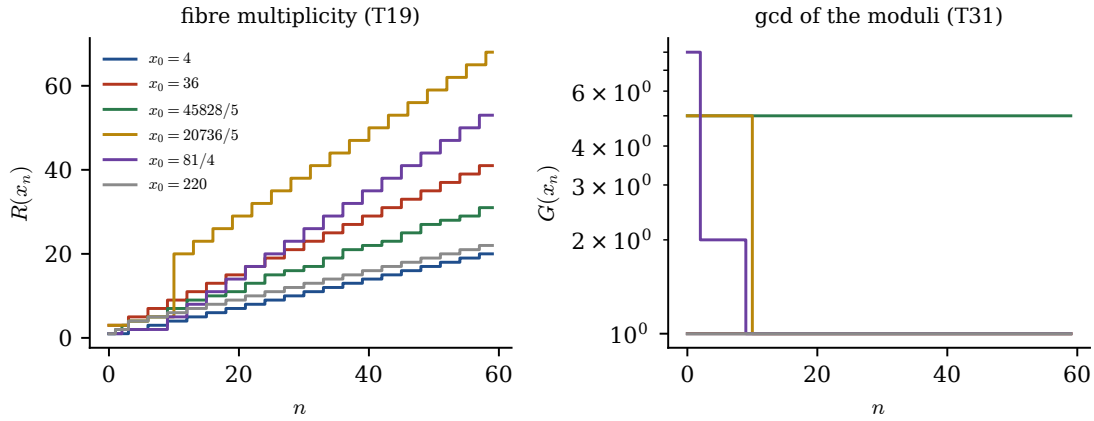


Figure 1.1: Collective growth along six exact orbits (60 steps each). Left: $R(x_n)$, non-decreasing and unbounded (Theorem 1.8). Right: $G(x_n)$, which can only drop through divisors (Theorem 1.3). The orbits from 4, 36, 220 have $G = 1$ throughout; 20736/5 (from $G = 5$) and 81/4 (from $G = 8$) collapse to $G = 1$; the orbit from 45828/5 stays at $G = 5$ for all 60 steps.

1.3 Unbounded generative multiplicity

Theorem 1.8 (Collective fibre growth). SOURCE RECORD: F.T19. *Scope C.* For every realised x_0 : $R(x_n)$ is non-decreasing, $R(x_n) \rightarrow \infty$, and $\bigcup_n \mathcal{D}(x_n)$ is infinite.

Proof. Monotonicity is Theorem 1.3. By Theorem 1.5 every growth event occurs within finitely many steps (for any $d \notin \mathcal{D}(x_n)$, which exists because $\mathcal{D}(x_n)$ is finite), and each event adds at least one modulus to a set that never shrinks. So there are infinitely many events and R is unbounded; being non-decreasing, it tends to infinity. $\square$

There is no infinite plateau, and no orbit has bounded fibre. The theorem makes no claim about *rate*; the next proposition gives an explicit but weak lower bound, and Chapters 2–4 determine the true rate in every regime (linear growth of R in the regimes found).

Proposition 1.9 (A provable growth rate). SOURCE RECORD: F.T27. *Scope C.* Let $v_0 = G(x_0)$. (a) $G(x_n) \leq v_0$ for all n . (b) $d_{\min}(x) \leq R(x) + 2$. (c) If $R(x_n) = R$ then $T(x_n) \leq v_0(R + 2)^2 - 1$, and

$$R(x_n) > \left(\frac{3n}{v_0}\right)^{1/3} - 3 \quad (n \geq 1).$$

Proof. (a) is Theorem 1.3. (b) The $R + 1$ integers $2, \dots, R + 2$ cannot all lie in a set of R elements. (c) Combine (a), (b) and Theorem 1.5. Let $n(r)$ be the first step with $R \geq r$. If $R(x_{n(r)}) = r$ then $n(r + 1) \leq n(r) + v_0(r + 2)^2 - 1$; otherwise $n(r + 1) = n(r)$. Therefore, if $R = R(x_n)$, then $n < n(R + 1) \leq v_0 \sum_{r=1}^R (r + 2)^2 \leq v_0(R + 3)^3/3$, which is the stated inequality. $\square$

The bound is deliberately weak (the sampled orbits of Figure 1.1 grow linearly), but it is unconditional, and it is the only rate available before the structure of Chapters 3–5 is in place.

1.4 Growth is specific to the collective branch

The monotonicity of the fibre is not a property of the arithmetic progression through a value; it is a property of the *collective* step, which advances by the common difference of the whole fibre.

Example 1.10 (Route-aware continuation does not preserve the fibre). **SOURCE RECORD:** F.T28. **Scope C.** Take $x = 36$ with the pair of routes $(2, 3)$, whose pairwise common-future step is $\Delta = 36$ (Volume I, Theorem 2.2). Along $x + k\Delta = 36(k + 1)$ the fibre sizes are

$$R(36(k + 1)) = 3, 3, 3, 5, 3, 3, 3, 5 \quad (k = 0, \dots, 7),$$

so R drops from 5 to 3 at 180, whereas along the collective orbit $\mathcal{D}$ never shrinks. (In the corpus, R decreases within ten steps along the pair orbit in every one of the 118 tested cases; this is data, not a claim.)

The two continuations differ for a precise reason: the collective step is a common multiple of every pairwise step, so each route continues to the new value (Theorem 1.3), while a pairwise step ignores the routes outside the pair and may leave their progressions.

Chapter 2 Arithmetic Regimes and the Evolution of Structural Invariants

The collective step is governed by the two invariants L and G of the fibre. This chapter studies how they evolve. We determine when $G = 1$, solve the dynamics in that regime in closed form, show that small values of G cannot persist, and derive the exact criterion by which a new modulus enters the fibre.

2.1 The square-root part and the arithmetic of G

Lemma 2.1 (Square-root reduction). *Write the numerator $a = s^2t$ of $x = a/b$ with t squarefree. Then $\gcd(s, b) = 1$ and*

$$\mathcal{D}(x) = \{d \mid s : d \geq 2, d \equiv 1 \pmod{b}\}.$$

Proof. $d^2 \mid a$ iff $2v_p(d) \leq v_p(a)$ for all p , i.e. $v_p(d) \leq \lfloor v_p(a)/2 \rfloor = v_p(s)$, i.e. $d \mid s$. $\gcd(a, b) = 1$ gives $\gcd(s, b) = 1$. (This is Volume I, Theorem 1.7.) $\square$

Theorem 2.2 (Arithmetic of G). **SOURCE RECORD:** F.T30. **Scope A0.** *For every realised $x = a/b$: (a) $b \mid G(x)$; (b) $\gcd(L, G) = 1$, so $\Delta_{\text{coll}} = L^2/G$ is reduced with denominator G ; (c)*

$$G(x) = 1 \iff 2 \in \mathcal{D}(x) \iff x \text{ is an integer divisible by 4.}$$

Proof. (a), (b) are Proposition 1.2. (c) If $2 \in \mathcal{D}$, then $4 \mid a$ and $2 \equiv 1 \pmod{b}$, so $b = 1$: x is an integer divisible by 4; and $G \mid 2 - 1 = 1$. Conversely, let x be an integer with $4 \mid x$, $x = s^2t$. Since $v_2(t) \leq 1$, $v_2(x) \geq 2$ forces $v_2(s) \geq 1$, so $2 \mid s$ and $2 \in \mathcal{D}$ by Lemma 2.1; hence $G = 1$. Finally if $G = 1$, then $b = 1$ by (a), and some $d \in \mathcal{D}$ is even (if all d were odd, every $d - 1$ would be even and $2 \mid G$); then $4 \mid d^2 \mid x$, so $2 \in \mathcal{D}$. $\square$

Thus $G = 1$ singles out the integers divisible by 4; every other realised value has $G \geq 2$. For $G > 1$ the whole fibre lies in a single residue class modulo G , which is why the arithmetic of residues governs Chapters 3–7.

2.2 Regime I: closed dynamics at $G = 1$

Theorem 2.3 (Regime I). SOURCE RECORD: F.T32. *Scope C.* Let $x = s^2 t$ be an integer divisible by 4, with t squarefree. Then s is even, $\mathcal{D}(x) = \{d \mid s : d \geq 2\}$, $G = 1$, $L = s$, $\Delta_{\text{coll}} = s^2$, $\gamma(x) = t$, and

$$C(x) = s^2(t+1) = s'^2 t', \quad s' = s\sigma, \quad t+1 = \sigma^2 t', \quad t' \text{ squarefree.}$$

In particular $C(x)$ is again an integer divisible by 4, so regime I persists. The squarefree parts follow $t \mapsto t' = \text{sqf}(t+1)$; every orbit of this map reaches the cycle $1 \rightarrow 2 \rightarrow 3 \rightarrow 1$ (with $\sigma = 1, 1, 2$), within at most $4\kappa_3(t)$ steps, where $\kappa_3(t)$ is the least k with $f^k(t) \leq 3$ and $f(a) = \lfloor a/4 \rfloor + 1$.

Proof. Since $4 \mid x$ and $v_2(t) \leq 1$, s is even. By Lemma 2.1, $\mathcal{D} = \{d \mid s, d \geq 2\} \ni s$; so $L = s$, and $G = 1$ by Theorem 2.2(c). Then $\Delta_{\text{coll}} = s^2$, $C(x) = s^2 t + s^2 = s^2(t+1)$, and $\gamma = x/s^2 = t$. Writing $t+1 = \sigma^2 t'$ gives the displayed form with $s' = s\sigma$ even, and $C(x)$ is an integer divisible by 4.

Absorption. Let $t_0 = t$, $t_{i+1} = \text{sqf}(t_i + 1)$. If $t \leq 3$ the sequence is $1 \rightarrow 2 \rightarrow 3 \rightarrow \text{sqf}(4) = 1$. Let $t \geq 4$ and choose $j \in \{1, 2, 3, 4\}$ with $4 \mid t+j$. As long as $t_i + 1$ is squarefree, $t_{i+1} = t_i + 1$, so $t_i = t + i$. Hence there is a first $i^* \leq j-1 \leq 3$ at which $t_{i^*} + 1 = t + i^* + 1$ is not squarefree, and then $t_{i^*+1} = \text{sqf}(t + i^* + 1) \leq (t + i^* + 1)/4 \leq (t + 4)/4$, i.e. $t_{i^*+1} \leq f(t)$, after at most four steps. Repeating, the values at the end of successive phases satisfy $a_{k+1} \leq f(a_k)$, hence $a_k \leq f^k(t)$ (f is nondecreasing), and no value ≤ 3 occurs inside a phase because those values exceed $t \geq 4$. So the cycle is entered after at most $4\kappa_3(t)$ steps. $\square$

Theorem 2.4 (Exact eventual three-step scaling in regime I). SOURCE RECORD: F.L3. *Scope C.* Let N be the entry time of Theorem 2.3, s_N the value of s then, and o the odd part of s_N . For $n \geq N$: $\gamma_n = t_n \in \{1, 2, 3\}$, $s_{n+3} = 2s_n$, $x_{n+3} = 4x_n$, $R(x_n) = \tau(s_n) - 1$ and

$$R(x_{n+3}) - R(x_n) = \tau(o).$$

There is no three-step law before N .

Proof. In the cycle $t = 1, 2, 3$ the factors σ are $1, 1, 2$, so over any three consecutive steps s is multiplied by 2 and t returns to itself; thus $x_{n+3} = 4x_n$. By Theorem 2.3, $R = \tau(s) - 1$ (divisors of s other than 1). If $s = 2^k o$, then $\tau(2s) - \tau(s) = \tau(o)(k+2) - \tau(o)(k+1) = \tau(o)$, and the odd part o never changes. $\square$

So in regime I the fibre grows linearly, $R(x_n) = \tau(o)(n/3 + O(1))$, the first instance of the structural growth that Chapters 4–5 extend to every orbit.

Theorem 2.5 (Exact reach counting at $G = 1$). SOURCE RECORD: F.L4. *Scope C.* After entry into regime I, the cumulative set of moduli is

$$U = \bigcup_n \mathcal{D}(x_n) = \{q2^k : q \mid o, k \geq 0, q2^k \geq 2\},$$

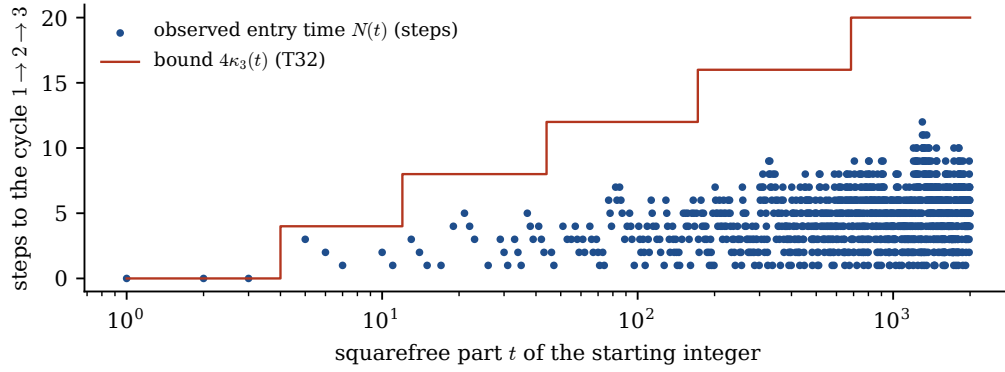


Figure 2.1: Entry time into the cycle $1 \rightarrow 2 \rightarrow 3$ of regime I for every squarefree $t \leq 2000$ (Theorem 2.3); the observed maximum is 12 steps, against the bound $4\kappa_3(t) \leq 20$.

and for every integer $X \geq 1$

$$U(X) = \#\{d \in U : d \leq X\} = \sum_{q|o, q \leq X} \left(1 + \lfloor \log_2(X/q) \rfloor\right) - 1.$$

Hence $U(X) = \tau(o) \log_2 X + O(1)$ and U has natural density zero.

Proof. By monotonicity $\mathcal{D}_n \subseteq \mathcal{D}_N$ for $n \leq N$, and $\mathcal{D}_N$ is the set of divisors ≥ 2 of $s_N = 2^{k_N}o$, contained in the displayed set. For $n \geq N$, $s_n = 2^{k_n}o$ with $k_n \rightarrow \infty$, and $\mathcal{D}_n$ is the set of divisors ≥ 2 of s_n . The union is the displayed set. The count is the number of pairs (q, k) with $q | o$, $q2^k \leq X$, minus the pair $(1, 0)$. $\square$

2.3 Modular escape and the first exclusions

A fibre cannot stand still for long when $G > 1$: the progression it follows eventually meets the integers divisible by 4, where G would be 1.

Lemma 2.6 (Modular escape). SOURCE RECORD: F.L1. *Scope C.* Let $G(x) = g > 1$ and suppose $\mathcal{D}$ is unchanged for steps $0, \dots, k$. Put $y_j = gx_j = y_0 + jL^2$. If L is odd, there is $k_* \in [1, 4g]$ with $y_{k_*} \equiv 0 \pmod{4g}$; if L is even (then g is odd), there is $k_* \in [1, g]$ with the same property. In either case $\mathcal{D}$ cannot be unchanged through step k_* .

Proof. If L is odd, $\gcd(L^2, 4g) = 1$ and $y_0 + jL^2 \equiv 0 \pmod{4g}$ is solvable with $1 \leq j \leq 4g$. If L is even, then g is odd (for even g every $d \equiv 1 \pmod{g}$ is odd), an even $d \in \mathcal{D}$ gives $4 | a$, hence $4 | y_0 = a(g/b)$, and $y_j = 4(y_0/4 + j(L/2)^2)$ with $\gcd((L/2)^2, g) = 1$; so $y_j \equiv 0 \pmod{4g}$ iff $y_0/4 + j(L/2)^2 \equiv 0 \pmod{g}$, solvable with $1 \leq j \leq g$. If $\mathcal{D}$ were unchanged through k_* , then $x_{k_*} = y_{k_*}/g$ would be an integer divisible by 4, so $G(x_{k_*}) = 1$ by Theorem 2.2(c), contradicting $G(x_{k_*}) = g > 1$. $\square$

Proposition 2.7 (Plateau escape). SOURCE RECORD: F.T37. *Scope C.* In an orbit with $G = g > 1$ at the start of a plateau, the plateau has at most $4g$ steps if L is odd and at most g steps if L is even. Consequently a persistent orbit (one with $G_n = g > 1$ for all large n) has infinitely many growth events, all G -preserving.

Proof. The first assertion is Lemma 2.6. If $G_n = g$ for all $n \geq n_0$, every plateau is finite, so there are infinitely many growth events; none reduces G , by persistence. $\square$

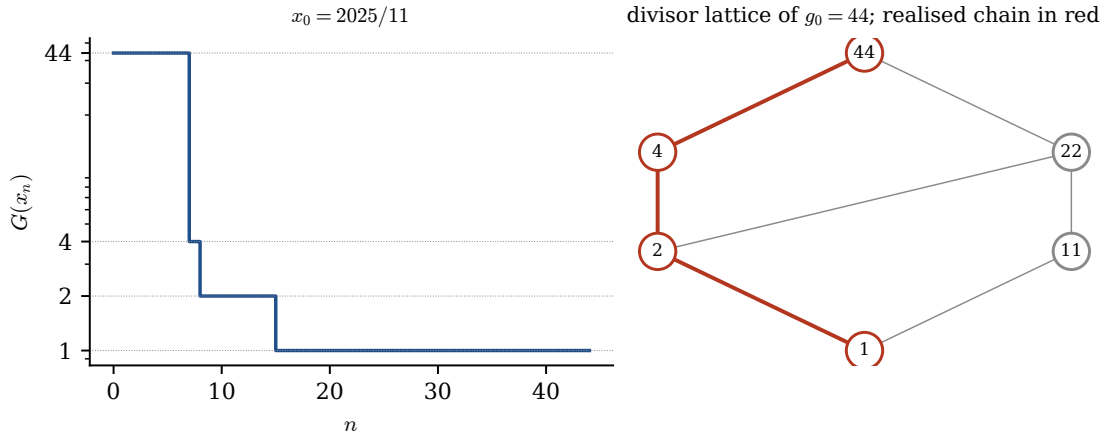


Figure 2.2: A divisor chain of G in an actual orbit. Left: $G(x_n)$ for $x_0 = 2025/11$ steps down $44 \rightarrow 4 \rightarrow 2 \rightarrow 1$. Right: the divisor lattice of $g_0 = 44$ with the realised chain in red. Each step of the chain is a reducing event, so $G_{n+1} \mid G_n$ (Theorem 1.3); the values 2 and 4 are transient by Theorems 2.8 and 3.13.

Theorem 2.8 (Collapse from $G = 2, 3, 6$). SOURCE RECORD: F.T34, F.L2. *Scope C.* If $G(x) = 2$ then $G(x_n) = 1$ for some $n \leq 7$. If $G(x) = 3$ then $G(x_n) = 1$ for some $n \leq 2$. If $G(x) = 6$ then $G(x_n) = 1$ for some $n \leq 29$. In particular none of 2, 3, 6 is the limit G_∞ of an orbit.

Proof. $G = 2$. All d are odd, so L is odd and $L^2 \equiv 1 \pmod{8}$. With $y = 2x$, $y' = y + L^2$, so while $G = 2$ the residue $y \pmod{8}$ increases by one per step, even across growth events. Initially $y \not\equiv 0 \pmod{8}$, since otherwise $x \in 4\mathbb{Z}$ and $G = 1$. Within seven steps $y \equiv 0 \pmod{8}$ would occur; at that state x is an integer divisible by 4, so $G = 1$. Since G can only decrease through divisors of 2, G reaches 1 within seven steps.

$G = 3$. If all d were odd, $2 \mid G$; so L is even and $4 \mid a$, hence $4 \mid y = 3x$. While $G = 3$, $y/4$ increases by $(L/2)^2 \equiv 1 \pmod{3}$ per step, starting from a nonzero residue (otherwise $x \in 4\mathbb{Z}$). Within two steps $y/4 \equiv 0 \pmod{3}$, i.e. $x = 4(y/4)/3$ is an integer divisible by 4 and $G = 1$.

$G = 6$. Every $d \equiv 1 \pmod{6}$ is coprime to 6, so $L^2 \equiv 1 \pmod{24}$, and $y = 6x$ increases by one modulo 24 per step while $G = 6$, starting nonzero. At the step $k^* \leq 23$ with $y \equiv 0 \pmod{24}$ the value x is an integer divisible by 4, so $G = 1$ there at the latest. If G first drops to 2 or 3, this happens at a step $j \leq 22$, after which at most 7 further steps are needed; so $G = 1$ within $22 + 7 = 29$ steps. $\square$

Theorem 2.8 removes three values; Theorem 3.13 of the next chapter removes every divisor of 24 and Theorem 3.2 shows that these are not the only obstructions that can occur.

2.4 Transitions between fibres

How does a new modulus enter the fibre? The following criterion gives the exact answer in terms of $y = Gx$.

Theorem 2.9 (Transition criterion). SOURCE RECORD: F.T35. *Scope C.* Let x be realised,

$g = G(x)$, $L = L(x)$, $y = gx$, $Y = y + L^2$, so that $C(x) = Y/g$. For every integer $d \geq 2$,

$$d \in \mathcal{D}(C(x)) \iff d^2 \mid Y \quad \text{and} \quad \frac{g}{\gcd(g, d-1)} \mid \frac{Y}{d^2}.$$

Consequently $G(C(x)) = \gcd(g, \{d-1 : d \in \mathcal{D}(C(x)) \setminus \mathcal{D}(x)\})$ (the gcd with the empty set is g). A step is therefore of exactly one of three kinds: none ($\mathcal{D}' = \mathcal{D}$); preserving ($\mathcal{D}' \supsetneq \mathcal{D}$, every new $d \equiv 1 \pmod{g}$, $G' = g$); reducing (some new $d \not\equiv 1 \pmod{g}$, $G' < g$). A modulus $d \equiv 1 \pmod{g}$ lies in $\mathcal{D}(C(x))$ iff $d^2 \mid Y$.

Proof. By Volume I (Theorem 1.2), $d \in \mathcal{D}(x')$ iff $x'(d-1)/d^2$ is a positive integer, i.e. $Y(d-1)/(gd^2) \in \mathbb{Z}_{>0}$. As $\gcd(d^2, d-1) = 1$, this forces $d^2 \mid Y$, and then it reads $g \mid (Y/d^2)(d-1)$, i.e. $g/\gcd(g, d-1) \mid Y/d^2$. The converse is the same chain. The formula for G' follows from $\mathcal{D} \subseteq \mathcal{D}'$: $G' = \gcd(G, \gcd\{d-1 : d \in \mathcal{D}' \setminus \mathcal{D}\})$. If $d \equiv 1 \pmod{g}$, then $\gcd(g, d-1) = g$ and the second condition is vacuous. $\square$

Theorem 2.10 (Anatomy of a preserving event). SOURCE RECORD: F.T36. *Scope C.* Let $x \rightarrow x'$ be a preserving event, $g = G(x)$, $y = gx$, $Y = y + L^2$, and $\mathcal{N} = \mathcal{D}(x') \setminus \mathcal{D}(x)$. Then:

- (a) every $d \in \mathcal{N}$ is coprime to g ;
- (b) no $d \in \mathcal{N}$ divides L ; thus each $d \in \mathcal{N}$ contains a fresh prime ($q \nmid L$) or a carried prime power ($v_q(d) > v_q(L) > 0$);
- (c) for a fresh prime $q \mid d$: $q \nmid y$, $q \nmid g$, and $y \equiv -L^2 \pmod{q^2}$, i.e. $-y$ is a nonzero square modulo q^2 ;
- (d) for a carried prime q , $l = v_q(L)$: $v_q(y) = 2l$ exactly, and $q \mid y/q^{2l} + (L/q^l)^2$;
- (e) $\mathcal{N}$ is closed under divisors in the residue class: if $d \in \mathcal{N}$, $e \mid d$, $e \geq 2$, $e \equiv 1 \pmod{g}$, then $e \in \mathcal{N} \cup \mathcal{D}(x)$.

Proof. (a) $d \equiv 1 \pmod{g}$. (b) If $d \mid L$ and $d \equiv 1 \pmod{g}$, then $d^2 \mid L^2 \mid y = a(g/b)$ and $\gcd(d, g) = 1$ give $d^2 \mid a$; also $d \equiv 1 \pmod{b}$ since $b \mid g$; so $d \in \mathcal{D}(x)$, contradicting newness. Hence some prime q has $v_q(d) > v_q(L)$, which is the fresh or carried case. (c) $d^2 \mid Y$ (Theorem 2.9) gives $q^2 \mid y + L^2$; $q \nmid L$ then forces $q \nmid y$, and $q \nmid g$ by (a). (d) $v_q(Y) \geq 2v_q(d) > 2l$ while $v_q(L^2) = 2l$ and $v_q(y) \geq 2l$ (Proposition 1.2(c)); the valuation of the sum exceeds $2l$ only if $v_q(y) = 2l$ and the q -free parts cancel modulo q . (e) $e^2 \mid d^2 \mid Y$ and $e \equiv 1 \pmod{g}$; Theorem 2.9 gives $e \in \mathcal{D}(x') = \mathcal{D} \cup \mathcal{N}$. $\square$

Reading. A preserving event is never an internal rearrangement of the present fibre. It introduces either a new prime, through an arithmetic coincidence $y \equiv -L^2 \pmod{q^2}$, or raises the exponent of a prime already present, a *carry*. Carries are the events that drive the whole theory of Chapters 4–7.

2.5 The reach set and non-universality

The union $U = \bigcup_n \mathcal{D}(x_n)$ is infinite (Theorem 1.8). It is nevertheless sparse.

Theorem 2.11 (Non-universality of reach). SOURCE RECORD: F.T33. *Scope C.* Let $G_\infty = \lim G_n$ (the limit exists: $G_{n+1} \mid G_n$).

- (A) If $G_\infty > 1$, then $U \subseteq \{d \geq 2 : d \equiv 1 \pmod{G_\infty}\}$; in particular $2 \notin U$, U contains no multiple of G_∞ , and U has upper asymptotic density at most $1/G_\infty$.

(B) If $G_\infty = 1$, the orbit is eventually in regime I and $U = \{q2^k : q \mid o, k \geq 0\} \setminus \{1\}$ is exactly as in Theorem 2.5; every d having an odd prime factor outside o is never reached, and $U(X) = O(\log X)$.

Hence for every orbit the reach set U omits infinitely many integers $d \geq 2$: not every index is ever reached, although the fibre grows without bound.

Proof. For every n , each $d \in \mathcal{D}_n$ satisfies $d \equiv 1 \pmod{G_n}$, hence modulo $G_\infty \mid G_n$. This gives (A). In case (B), $G_{n_0} = 1$ for some n_0 , so x_{n_0} is an integer divisible by 4 (Theorem 2.2), regime I applies (Theorem 2.3), and Theorem 2.5 gives U ; moduli at earlier steps lie in $\mathcal{D}_{n_0}$ by monotonicity. For the counting: $U(X) \leq \tau(o)(1 + \log_2 X)$. $\square$

For instance the orbit of 4 is 4, 8, 12, 16, 32, 48, 64, 128, ... and reaches only powers of two.

Remark 2.12 (Scope of the density statement). SOURCE RECORD: F.L5. Scope C. Part (A) proves, for $G_\infty > 1$, the upper density bound $1/G_\infty$ and the exclusion of infinitely many moduli. The logarithmic counting function and density zero are proved above for $G_\infty = 1$. That they hold for *every* orbit, including those with $G_\infty > 1$, is a consequence of single-prime dominance and is proved in Corollary 5.7. In this form non-universality is established for all collective orbits and the density statement is stated exactly where it is proved.

Chapter 3 Persistence, Admissibility, and Structural Exclusion

Does G always collapse to 1? If it did, regime I would be the whole story. It does not. This chapter introduces the integer coordinate γ that makes persistent orbits tractable, constructs persistent regimes for infinitely many g , shows that universal collapse is false, and proves that certain moduli can never persist.

3.1 The γ -coordinate

For a realised x with $g = G(x)$ and $L = L(x)$, Proposition 1.2 gives $x = \gamma L^2/g$ with $\gamma \in \mathbb{Z}_{>0}$. This coordinate evolves by simple rules.

Theorem 3.1 (The γ -framework). SOURCE RECORD: F.T44. *Scope C. Let x be realised, $g = G(x)$, $L = L(x)$, $\gamma = \gamma(x)$.*

- (a) $x = \gamma L^2/g$ with $\gamma \in \mathbb{Z}_{>0}$, $b \mid g$, $\gcd(L, g) = 1$.
- (b) The number $\gamma L^2/g$ is an integer divisible by 4 iff $4g \mid \gamma$ (L odd) or $g \mid \gamma$ (L even, g odd). (For $g > 1$ this cannot occur at a state of the orbit: it would force $G = 1$.)
- (c) If the step $x \rightarrow x'$ is G -preserving and $r = L(x')/L(x) \in \mathbb{Z}_{\geq 1}$, then $\gamma(x') = (\gamma + 1)/r^2$. On a plateau $\gamma' = \gamma + 1$.
- (d) (Congruence advance) If g is even, then r is odd and $\gamma' \equiv \gamma + 1 \pmod{8}$. If $3 \mid g$, then $\gamma' \equiv \gamma + 1 \pmod{3}$. If $6 \mid g$, then $\gamma' \equiv \gamma + 1 \pmod{24}$.
- (e) The reduced denominator of x is $b = g/\gcd(g, \gamma)$; so $g \mid \gamma$ iff x is an integer.

Proof. (a) is Proposition 1.2. (b) $\gcd(L, g) = 1$. If L is odd, $\gcd(L^2, 4g) = 1$ and $4g \mid \gamma L^2$ iff $4g \mid \gamma$. If L is even then g is odd (for even g all d are odd), $L^2 = 4(L/2)^2$ with $\gcd(L/2, g) = 1$,

and $4g \mid 4\gamma(L/2)^2$ iff $g \mid \gamma$. If $G(x) > 1$ the conclusion $G = 1$ of Theorem 2.2(c) is impossible. (c) With $G' = g$, $gx' = y + L^2$ and $\gamma' = gx'/L^2 = (\gamma + 1)L^2/L^2 = (\gamma + 1)/r^2$. (d) $\gamma'r^2 = \gamma + 1$. If g is even, every modulus is odd, so r is odd and $r^2 \equiv 1 \pmod{8}$. If $3 \mid g$, all moduli are $\equiv 1 \pmod{3}$, so $3 \nmid r$ and $r^2 \equiv 1 \pmod{3}$; if $6 \mid g$, r is prime to 6 and $r^2 \equiv 1 \pmod{24}$. (e) $\gcd(L^2, g) = 1$, so the reduced denominator of $\gamma L^2/g$ is $g/\gcd(g, \gamma)$. $\square$

Call an orbit *persistent* if $G_n = g > 1$ for all $n \geq n_0$; since G_n is non-increasing through divisors, G_∞ is attained, and every orbit with $G_\infty > 1$ is eventually persistent. In a persistent orbit every growth event is G -preserving (Theorem 2.9), so Theorem 3.1(c) applies at every step: the orbit is the pair (γ_n, L_n) , with γ advancing by 1 on plateaus and dropping to $(\gamma + 1)/r^2$ at carries.

3.2 Persistent regimes

Theorem 3.2 (The regime P_g). SOURCE RECORD: F.T38. *Scope C.* Let $g \geq 5$ with $\gcd(g, 6) = 1$ and $s = 2^k m$ with $k \geq 1$, m odd, $\gcd(m, g) = 1$. Assume

- (H1) the residues modulo g of the divisors of m contain the subgroup $\langle 2 \rangle \leq (\mathbb{Z}/g)^\times$;
- (H2) every prime power $q^e \parallel m$ divides some divisor d of $2^k m$ with $d \equiv 1 \pmod{g}$;
- (H3) $\gcd\{(d-1)/g : d \mid s, d \geq 2, d \equiv 1 \pmod{g}\} = 1$.

Put $x_0 = s^2 t/g$ with $t \in \{1, 2, 3\}$, and write $s_n = s \cdot$ (product of the factors σ), $t_{n+1} = \text{sqf}(t_n + 1)$ with $\sigma_n^2 \parallel t_n + 1$ ($\sigma = 2$ exactly when $t_n = 3$, else 1). Then for every $n \geq 0$:

- (i) $x_n = s_n^2 t_n/g$ in lowest terms, $t_n \in \{1, 2, 3\}$;
- (ii) $\mathcal{D}(x_n) = \{d \mid s_n : d \geq 2, d \equiv 1 \pmod{g}\}$, $L(x_n) = s_n$, $G(x_n) = g$;
- (iii) $x_{n+3} = 4x_n$, and $G(x_n) = g$ for all n : the orbit never reaches $G = 1$.

Proof. Induction on n . Suppose (i) at n , with $2^{k_n} \parallel s_n$, $k_n \geq k$, and odd part m (unchanged, since $\sigma \in \{1, 2\}$). Because g is prime to 6, $\gcd(s, g) = \gcd(t, g) = 1$, so $x_n = s_n^2 t_n/g$ is reduced with $b = g$. As t_n is squarefree, the square-root part is s_n and, by Lemma 2.1, $\mathcal{D}(x_n) = \{d \mid s_n : d \geq 2, d \equiv 1 \pmod{g}\}$. $L = s_n$: for the 2-part, (H1) gives $u \mid m$ with $u \equiv 2^{-k_n} \pmod{g}$, so $d = 2^{k_n} u \in \mathcal{D}$; for $q^e \parallel m$, (H2) gives a modulus divisible by q^e (it divides s_n because s_n is a multiple of $2^k m$). $G = g$: $g \mid G$ as every $d \equiv 1 \pmod{g}$; $G(x_n) \mid G(x_0)$ by Theorem 1.3; and $G(x_0) = g \gcd\{(d-1)/g\} = g$ by (H3). Hence $\Delta_{\text{coll}} = s_n^2/g$ and

$$x_{n+1} = \frac{s_n^2 t_n + s_n^2}{g} = \frac{s_n^2 (t_n + 1)}{g} = \frac{s_{n+1}^2 t_{n+1}}{g},$$

with $t_{n+1} \in \{2, 3, 1\}$ prime to g . So (i) and (ii) hold at $n+1$. Over three steps the factors σ are 1, 1, 2 in some phase, so s doubles and t returns: $x_{n+3} = 4x_n$. $\square$

Proposition 3.3 (Existence for every g prime to 6). SOURCE RECORD: F.T38A. *Scope C.* Let $g \geq 5$, $\gcd(g, 6) = 1$, q the least odd prime with $q \equiv 2 \pmod{g}$ (Dirichlet), $o = \text{ord}_g(2)$, $g+1 = 2^v w$ with w odd, $m = wq^o$, $k = \max(v, 1)$. Then (H1)–(H3) hold for $s = 2^k m$.

Proof. $\gcd(w, g) = 1$ as $w \mid g+1$, and $\gcd(q, g) = \gcd(2, g) = 1$. A prime divisor of $g+1$ is at most $g+1$, and a prime $\equiv 2 \pmod{g}$ at most $g+1$ is 2; so the odd prime q does not divide $g+1$, and $q^o \parallel m$. (H1): the divisors $q^i \equiv 2^i \pmod{g}$ ($0 \leq i < o$) exhaust $\langle 2 \rangle$. (H2): $q^o \equiv 2^o = 1 \pmod{g}$ is a modulus divisible by q^o ; each prime power of w divides $g+1 \mid s$, and $g+1 \equiv 1$. (H3): $d = g+1$ has $(d-1)/g = 1$. $\square$

The Dirichlet step can be avoided altogether.

Theorem 3.4 (Elementary construction). SOURCE RECORD: F.T42. *Scope C.* Let $g \geq 5$, $\gcd(g, 6) = 1$, $o = \text{ord}_g(2)$, $s = (g+1)(g+2)^o$ and $x_0 = s^2/g$. Then $x_n = s_n^2 t_n / g$ with $t_n \in \{1, 2, 3\}$, $G(x_n) = g$ and $x_{n+3} = 4x_n$ for all n .

Proof. Let $k = v_2(g+1) \geq 1$ and $m = s/2^k = \text{odd}(g+1)(g+2)^o$, odd and prime to g (g is odd and $\gcd(g+1, g) = \gcd(g+2, g) = 1$). We verify the hypotheses of Theorem 3.2. (H1): the divisors $(g+2)^j$ of m are $\equiv 2^j \pmod{g}$, so their residues contain $\langle 2 \rangle$. (H2): since $\gcd(g+1, g+2) = 1$, a prime power $q^e \parallel m$ lies in $g+1$ (and $g+1 \equiv 1$ is a modulus dividing s) or in $(g+2)^o$, which is itself a modulus: $(g+2)^o \equiv 2^o = 1$. (H3): $d = g+1$ gives $(d-1)/g = 1$. $\square$

Instances. Direct iteration of C confirms Theorem 3.4 for $g = 5, 7, 11, 13, 17, 19, 23, 25, 29, 35, 49, 55, 65, 77$ (twelve steps each, in exact rational arithmetic, together with the hypotheses (H1)–(H3)). For $g = 5$: $o = 4$, $s = 6 \cdot 7^4 = 14406$, and $R(x_0) = \#\{d \mid s : d \geq 2, d \equiv 1 \pmod{5}\}$.

Theorem 3.5 (Persistent regimes with $3 \mid g$). SOURCE RECORD: F.T46. *Scope C.* Let $g \geq 5$ be odd ($3 \mid g$ allowed), $s = 2^k m$ with m odd, $\gcd(m, g) = 1$, $o = \text{ord}_g(2)$, and assume (H1)–(H3) of Theorem 3.2 and

(H4) for $t \in \{1, 2, 3\}$ and $b_t = g/\gcd(g, t)$, every $d = 2^j a$ ($a \mid m$, $j \geq 0$) with $d \equiv 1 \pmod{b_t}$ satisfies $d \equiv 1 \pmod{g}$.

Then for $x_0 = s^2/g$ the conclusions (i)–(iii) of Theorem 3.2 hold, and so does the periodic increment of Theorem 3.7.

Proof. As for Theorem 3.2; the single new point is that $x_n = s_n^2 t_n / g$ has reduced denominator $b_t = g/\gcd(g, t)$ (because $\gcd(s, g) = 1$ and t is squarefree), so $\mathcal{D}(x_n) = \{d \mid s_n : d \equiv 1 \pmod{b_t}\}$, which equals $\{d \mid s_n : d \equiv 1 \pmod{g}\}$ by (H4) (the residue of d depends only on $j \pmod{o}$). Then $G(x_n) \mid G(x_0) = g$, $b \mid G$, and $g \mid G$ give $G = g$ as before. $\square$

Instances. $g = 15, 39, 51, 57, 87$, with $s = (g+1)(g+2)^o$ (direct iteration, 15 steps each; (H4) checked by exhaustion). So $G_\infty > 1$ occurs with $\gcd(G_\infty, 6) = 3$.

Remark 3.6 (Why $3 \mid g$ works and $2 \mid g$ does not). For $g = 15$ and $t = 3$ the denominator is $b = 5$, and the divisors of s lie in $\langle 2 \rangle \pmod{15}$, on which reduction modulo 5 is injective ($\text{ord}_5(2) = \text{ord}_{15}(2) = 4$); so no new modulus appears. For even g , every modulus $d \equiv 1 \pmod{g}$ is odd, powers of 2 are never moduli, and $L \neq s$; the (s, t) -description of Theorem 3.2 breaks down. Even g is governed instead by the carry structure of Chapter 4.

Theorem 3.7 (Periodic increment of the fibre). SOURCE RECORD: F.T43. *Scope C.* In any orbit satisfying Theorem 3.2 or 3.5, let $s_n = 2^{k_n} m$ (m odd, fixed), $o = \text{ord}_g(2)$, and $A_g(m) = \#\{a \mid m : a \pmod{g} \in \langle 2 \rangle\}$. Then

$$R(x_n) = \#\{(a, j) : a \mid m, 0 \leq j \leq k_n, 2^j a \equiv 1 \pmod{g}\} - 1, \quad k_{n+3o} = k_n + o,$$

and $R(x_{n+3o}) - R(x_n) = A_g(m)$ for every n .

Proof. $\mathcal{D}(x_n) = \{d \mid s_n : d \geq 2, d \equiv 1 \pmod{g}\}$, and $d = 2^j a$ uniquely ($a \mid m$ odd, $j \leq k_n$); the -1 removes $d = 1$. Every three steps s is multiplied by exactly 2, so $k_{n+3o} = k_n + o$. The new pairs have $j \in (k_n, k_n + o]$. Since $2^j \pmod{g}$ has exact period o , over o consecutive values of j the residue 2^j meets each element of $\langle 2 \rangle$ exactly once; so for each a with $a \pmod{g} \in \langle 2 \rangle$ exactly one j gives $2^j a \equiv 1$, and none for other a . $\square$

Theorem 3.8 (Reach set and growth in P_g). SOURCE RECORD: F.T40. *Scope C.* In the orbits of Theorems 3.2 and 3.5 the reach set is $U = \{d \geq 2 : d \mid 2^k m \text{ for some } k, d \equiv 1 \pmod{g}\}$. It contains no multiple of g and none of $2, 3, \dots, g$, and $U(X) = O(\log X)$ (density zero), while $R(x_n)$ grows linearly with slope $A_g(m)/(3o)$ per step.

Proof. The union of the sets $\mathcal{D}(x_n)$ over n (with $k_n \rightarrow \infty$) is the displayed set. A modulus $\equiv 1 \pmod{g}$ and ≥ 2 is at least $g + 1$. The count is at most $\tau(m)(1 + \log_2 X)$; the slope is Theorem 3.7. $\square$

Example 3.9 (A natural seed). The orbit of $x_0 = 45828/5 = 36 \cdot 1273/5$ has $\mathcal{D} = \{6\}$, $G = 5$, $\gamma_0 = 1273$ (Example 1.7). From step 3 on it is in the cycle $\gamma = 1, 2, 3$ with L doubling every three steps; the state $x_3 = 63504/5$ is a reset state whose class cycle closes (Theorem 4.5), so $G_n = 5$ for all n and $x_{n+3} = 4x_n$ for $n \geq 3$. The orbit of $20736/5$, by contrast, has $G = 5$ for nine steps and then reaches $G = 1$ (Figure 1.1).

3.3 Universal collapse is false

Theorem 3.10 (Universal G -collapse is false). SOURCE RECORD: F.T39. *Scope C.* For every $g \geq 5$ with $\gcd(g, 6) = 1$ there are collective orbits with $G(x_n) = g > 1$ for every n . Hence the statement “if $G_n > 1$ then $G_k < G_n$ for some $k > n$ ” is false, and the question whether every orbit reaches $G = 1$ is answered in the negative.

Proof. Theorem 3.4 (or Proposition 3.3). $\square$

Remark 3.11 (Resolution of the collapse problem). SOURCE RECORD: F.L6. *Scope C.* Lemma 2.6 prevents unbounded plateaus and Theorem 2.8 removes $G = 2, 3, 6$, but neither excludes infinitely many G -preserving growth events. The collapse question, whether every realised initial value reaches $G = 1$, is therefore settled by Theorem 3.10: it does not. Random sweeps of natural seeds miss the phenomenon: among $a < 25000$ with $G = 5$ only 10 of 557 persist, and no other g appears.

Remark 3.12 (Which g can persist). SOURCE RECORD: F.T41. *Scope C.* Known so far: $G_\infty \neq 2, 3, 6$ (Theorem 2.8); no divisor of 24 is a persistent value (Theorem 3.13 below); persistence occurs for every $g \geq 5$ prime to 6 (Theorem 3.4), for $g = 15, 39, 51, 57, 87$ (Theorem 3.5), and for the even value $g = 68$ (Example 4.13). The earlier expectation that persistence requires $\gcd(g, 6) = 1$ is superseded by Theorem 3.5. A complete classification of the g that admit persistent orbits is open (Appendix F).

3.4 Exclusion of $G_\infty \mid 24$

Theorem 3.13 (G_∞ cannot divide 24). SOURCE RECORD: F.T45. *Scope C.* If $g > 1$ divides 24, no collective orbit is persistent at $G = g$. In particular $G_\infty \notin \{2, 3, 4, 6, 8, 12, 24\}$.

Proof. Suppose the orbit is persistent at g from n_0 .

$g = 3$. L is even, G collapses iff $3 \mid \gamma$ (Theorem 3.1(b),(e)), and γ advances by 1 modulo 3 at every step (Theorem 3.1(d)); so $3 \mid \gamma$ within three steps, a contradiction.

g even. Then $g \mid 24$, $g \neq 3$, L is odd, and G collapses at the modulus $M = 4g$. Every prime of every r is prime to g .

Step 1 (prime classes). Let $n > n_0$ and $q \mid L_n$ a prime. By Theorem 3.1(d), γ advances by one modulo 8 if $g \mid 8$, or modulo 24 if $g \mid 24$ with $3 \nmid g$; in both cases some $m \geq n$ has $g \mid \gamma_m$ (since $g \mid 8$ or $g \mid 24$). By Theorem 3.1(e) x_m is an integer. Because $q \mid d_0$ for some $d_0 \in \mathcal{D}_n \subseteq \mathcal{D}_m$, $q^2 \mid a_m$, and since $b_m = 1$, $q \in \mathcal{D}(x_m)$. Persistence gives $g = G(x_m) \mid q - 1$: $q \equiv 1 \pmod{g}$.

Step 2 (growth is large). Every growth ratio r at a step after n_0 has all prime factors $\equiv 1 \pmod{g}$ by Step 1, so $r \geq g + 1$ and $r^2 \geq (g + 1)^2 > 4g = M$.

Step 3 (descent). A plateau has at most $M - 1$ states (otherwise $M \mid \gamma$, i.e. $4g \mid \gamma$, forcing $G = 1$); so growth events are infinite. Let a_k be the value of γ right after the k -th growth. The plateau runs $a_k, \dots, a_k + j$ avoid multiples of M , so $j \leq M - 2$ and the next growth carries $\gamma_{\text{before}} \leq a_k + M - 2$ to

$$a_{k+1} = \frac{\gamma_{\text{before}} + 1}{r^2} \leq \frac{a_k + M - 1}{(g + 1)^2} < a_k,$$

the last inequality because $M - 1 < a_k((g + 1)^2 - 1)$ (as $a_k \geq 1$ and $(g + 1)^2 - 1 = g^2 + 2g \geq 4g$). A strictly decreasing infinite sequence of positive integers does not exist. $\square$

The theorem reproves Theorem 2.8. For $g = 4$ the mechanism is that every prime of L must be $\equiv 1 \pmod{4}$, forcing $r \geq 5$ and $25 > 16$; for $g = 12$, every prime of L must be $\equiv 1 \pmod{12}$. The factor $r^2 \equiv 1 \pmod{8}$ is the reason the modular escape of Lemma 2.6 alone does not settle even g ; the descent argument replaces it.

3.5 A bounded window for γ

Theorem 3.14 (Persistent orbits live in a bounded γ -window). SOURCE RECORD: F.T47. *Scope C.* Let a persistent orbit have collapse modulus M ($M = 4g$ if g is even, $M = g$ if g is odd). Then $\gamma_n \leq 2M$ for all sufficiently large n .

Proof. (For odd g , L is even at every state, Theorem 3.15; for even g , L is odd.) Plateaus have at most $M - 1$ states (else $\gamma \equiv 0 \pmod{M}$ and $G = 1$, Theorem 3.1(b)), and every growth has $r \geq 2$, so $a_{k+1} \leq (a_k + M - 1)/4$. Then $a_k - \frac{M-1}{3} \leq 4^{-k}(a_0 - \frac{M-1}{3})$, so $\limsup a_k \leq (M - 1)/3$, and $\gamma_n \leq a_k + M - 2 < 2M$ eventually. $\square$

Meaning. Up to a bounded factor, a persistent state is the square of the lcm: $x_n = \gamma_n L_n^2 / g$ with γ_n in a finite window. A persistent orbit is a finite-state γ -process driven by arithmetic coincidences; in the regimes of this chapter the process is the cycle 1, 2, 3.

Lemma 3.15 (Parity of L in a persistent orbit). *If $g = G(x)$ is odd then $L(x)$ is even. If g is even then $L(x)$ is odd.*

Proof. If g is odd, some $d - 1$ is odd (else $2 \mid G$), so some $d \in \mathcal{D}$ is even. If g is even, every $d \equiv 1 \pmod{g}$ is odd. $\square$

Chapter 4 The Gamma Framework and Exact Asymptotic Scaling

Chapter 3 reduced a persistent orbit to the pair (γ_n, L_n) . This chapter makes that reduction exact: a one-step law computes the next fibre from γ and the exponents alone (Theorem 4.2); a finite-state closure shows that every persistent orbit is eventually class-periodic and therefore scales exactly (Theorem 4.5); and the carry mechanism is analysed until its structure is explicit (Theorems 4.6–4.12). Throughout, $x = \gamma L^2/g$ is a persistent state with $G(x) = g > 1$, $L = \prod_p p^{e_p}$, $o_p = \text{ord}_g(p)$, $w = \gamma + 1$, $h = \gcd(g, w)$, $b' = g/h$. A *carry* is a step with $L_{n+1} \neq L_n$, $r = L_{n+1}/L_n$; a *plateau* is a run of steps with $r = 1$.

Remark 4.1 (The scaling question and its answer). SOURCE RECORD: F.T48. Scope C. The natural conjecture, that every orbit eventually satisfies $x_{n+3} = 4x_n$ with $x_n = s_n^2 t_n/g$, $t_n \in \{1, 2, 3\}$, holds for $G_\infty = 1$ (Theorem 2.3) and for the regimes of Theorems 3.2 and 3.5, but it is not a statement about all orbits: the orbit of Theorem 4.3 below has x_{n+3}/x_n taking eight different values forever. The correct universal statement is Theorem 4.5: every orbit eventually satisfies $x_{n+P} = \rho^2 x_n$ for some period P and some integer $\rho \geq 2$, with (P, ρ) depending on the orbit.

4.1 The one-step law

Theorem 4.2 (One-step law). SOURCE RECORD: F.T49. Scope C. Let $x = \gamma L^2/g$ be persistent. Put $\sigma' = L \prod_p p^{\lfloor v_p(w/h)/2 \rfloor}$. Then $x' = C(x) = wL^2/g$ and

$$\mathcal{D}(x') = \{d \mid \sigma' : d \geq 2, d \equiv 1 \pmod{b'}\}.$$

The point x' is persistent iff every d in this set is $\equiv 1 \pmod{g}$. In that case $L' = \text{lcm } \mathcal{D}(x')$ is a multiple of L , $r = L'/L$ is an integer with $r \mid \prod_p p^{\lfloor v_p(w/h)/2 \rfloor}$ and $r^2 \mid w$, and

$$\gamma' = w/r^2.$$

Proof. $x' = x + L^2/g = (\gamma + 1)L^2/g = wL^2/g$. As $\gcd(L, g) = 1$, the reduced form has numerator $(w/h)L^2$ and denominator $b' = g/h$. Now $d^2 \mid (w/h)L^2$ iff $2v_p(d) \leq v_p(w/h) + 2e_p$ for all p , iff $v_p(d) \leq e_p + \lfloor v_p(w/h)/2 \rfloor$, iff $d \mid \sigma'$. If every such $d \equiv 1 \pmod{g}$ then $g \mid G(x')$; and $G(x') \mid G(x) = g$ by Theorem 1.3; hence $G(x') = g$. Conversely persistence requires $g \mid d - 1$ for all $d \in \mathcal{D}(x')$. $L \mid L'$ from $\mathcal{D}(x) \subseteq \mathcal{D}(x')$, and $L' \mid \sigma'$ gives $r \mid \prod_p p^{\lfloor v_p(w/h)/2 \rfloor}$. Finally $\gamma' = gx'/L'^2 = wL^2/L'^2 = w/r^2$, an integer by Theorem 3.1(a). $\square$

The theorem makes the dynamics a computation on residues: the only information about the exponent e_p that enters is the set $\{p^k \bmod g : 0 \leq k \leq e_p + s'_p\}$. This is the “residue machine” used for all certificates in this volume (Appendix C).

4.2 The eight-step orbit and the failure of a uniform period

Theorem 4.3 (The eight-step orbit at $g = 11$). SOURCE RECORD: F.T50. Scope C. Let $S_k = 2^2 \cdot 3^{3+k} \cdot 5^4$ and $x = tS_k^2/11$. For every $k \geq 0$:

- (i) $L(tS_k^2/11) = S_k$ for $t = 1, \dots, 8$; $L(9S_k^2/11) = 3S_k = S_{k+1}$; and $G = 11$ throughout;

- (ii) the orbit of C from $x_0 = 67500^2/11$ is $x_{8k+j} = (j+1)S_k^2/11$ ($j = 0, \dots, 7$); hence $x_{n+8} = 9x_n$, with γ -cycle $1 \rightarrow 2 \rightarrow \dots \rightarrow 8 \rightarrow 1$ and one 3-carry per 8 steps;
- (iii) $R(x_{8k+j}) = 7 + 2k$, so $R(x_{n+8}) = R(x_n) + 2$;
- (iv) x_{n+3}/x_n takes the eight values $3/2, 8/5, 7/4, 2, 5/2, 18/7, 27/8, 4$, each once per period.

Proof. $g = 11$ is prime and $w \leq 9 < 11$, so $h = 1, b' = 11$. Modulo 11: 3 has order 5 and $H = \langle 3 \rangle = \{1, 3, 4, 5, 9\}$ (the quadratic residues); $2 \notin H, 4 = 3^4 \in H, 8 = 2 \cdot 4 \notin H, 5 \equiv 3^3, 12 \equiv 1$. By Theorem 4.2 the fibre of $wS_k^2/11$ is $\{d \mid S_k \cdot \text{sqrtpart}(w) : d \equiv 1 \pmod{11}\}$.

L is attained. Write $d = 2^a 3^b 5^c$; its residue is $2^a 3^{b+3c}$. Since $2 \notin H$ and $4 \in H$, the residue lies in H only for $a \in \{0, 2\}$ (for $a \leq 2$), and $d \equiv 1$ iff $5 \mid b + 3c$ ($a = 0$) or $5 \mid 4 + b + 3c$ ($a = 2$). For $b = 3 + k$ the congruence $c \equiv -(3 + k)3^{-1} \pmod{5}$ has a solution $c \in \{0, \dots, 4\}$, so $v_3 = 3 + k$ is attained. $5^4 3^i = 3^{12+i} \equiv 1$ needs $i \equiv 3 \pmod{5}$: $i = 3 \leq 3 + k$, so $v_5 = 4$ is attained; a suitable (b, c) with $a = 2$ attains $v_2 = 2$. These congruences are periodic in $k \pmod{5}$, so $L(S_k^2/11) = S_k$; for $t \in \{1, 2, 3, 5, 6, 7\}$, $\sigma' = S_k$, so L cannot grow.

No absorption for $t = 1, \dots, 8$. For $t \in \{4, 8\}$, $\sigma' = 2S_k$ has $v_2 = 3$; a divisor with $v_2 = 3$ would need $2^3 3^i 5^c \equiv 1$, i.e. $8 \in H$, which is false.

The 3-carry at $t = 9$. $\sigma' = 3S_k$ has $v_3 = 4 + k$; $d = 3^{4+k} 5^c$ with $5 \mid 4 + k + 3c$ is solvable, so $L' = 3S_k$ and $\gamma' = 9/9 = 1$.

$G = 11$. $12 = 2^2 \cdot 3 \mid S_k$ and $12 \equiv 1 \pmod{11}$ with $12 - 1 = 11$, so $G \mid 11$; every $d \equiv 1 \pmod{11}$, so $G = 11$.

Counts. $d = 2^a 3^b 5^c \equiv 1$ forces $a \in \{0, 2\}$; for each a and each c exactly one residue class of $b \pmod{5}$ works, and the classes for $c = 0, \dots, 4$ are distinct, so $\sum_c \#\{b \leq B : b \equiv b_c\} = B + 1$ with $B = 3 + k$. Hence $R = 2(4 + k) - 1$, the -1 removing $d = 1$. Item (iv) is exact arithmetic on the closed form of (ii). $\square$

The orbit of Theorem 4.3 is the first of a family with arbitrarily large cycle $1, \dots, q^2 - 1$ (Theorem 4.12); it shows concretely that the period 3 of regime I is a feature of $G_\infty = 1$, not of persistence as such (Remark 4.1).

4.3 Finite-state closure and exact scaling

For a prime $p \nmid g$ and $e \geq 0$ define the *class*

$$c_p(e) = \begin{cases} e, & e \leq o_p - 2, \\ (o_p - 1) + ((e - (o_p - 1)) \bmod o_p), & e \geq o_p - 1, \end{cases}$$

so classes are $0, 1, \dots, 2o_p - 2$. The class of a persistent state is $(\gamma, (c_p(e_p))_p)$, with $c_p = 0$ when $e_p = 0$.

Lemma 4.4 (Class lemma). *The class of $C(x)$, the carry vector $\Delta_p = e'_p - e_p$, and persistence of $C(x)$ are functions of the class of x .*

Proof. By Theorem 4.2 the data of the step are $w, h, b', s'_p = \lfloor v_p(w/h)/2 \rfloor$ (functions of γ) and, for each p , the residue set $\{p^k \bmod g : 0 \leq k \leq e_p + s'_p\}$. If $e_p \geq o_p - 1$, that set is all of $\langle p \rangle$ whatever e_p is; if $e_p < o_p - 1$ it depends on e_p exactly. The new exponent is $e_p + \Delta_p$ with

$$\Delta_p = \max\{j \in [0, s'_p] : p^{e_p+j} a \equiv 1 \pmod{b'} \text{ for some } a \in \mathcal{A}_{-p}\},$$

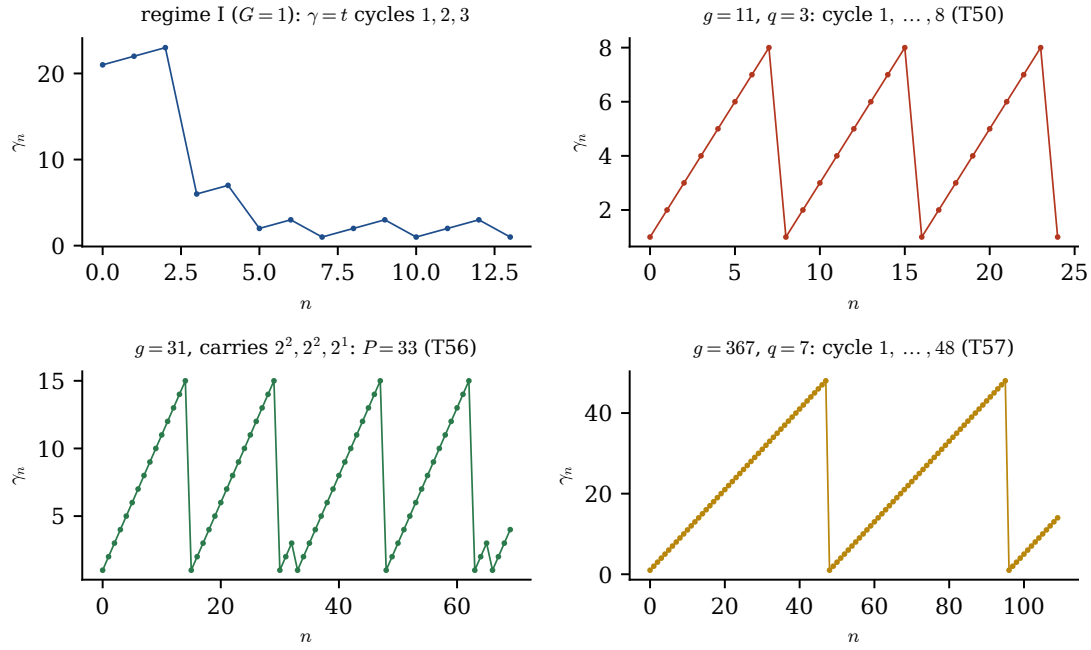


Figure 4.1: The arithmetic state cycles γ_n of four orbits, computed from the residue machine and cross-checked against direct iteration of C : regime I ($G = 1$, cycle 1, 2, 3); $g = 11$, $q = 3$ (cycle 1, ..., 8, Theorem 4.3); $g = 31$ with carries $2^2, 2^2, 2^1$ (period 33, Theorem 4.10); $g = 367$, $q = 7$ (cycle 1, ..., 48, Theorem 4.12). Each reset to $\gamma = 1$ is a carry.

where $\mathcal{A}_{-p}$ is the set of residues of divisors of $\sigma'/p^{v_p(\sigma')}$ ($j = 0$ is always attained since $\mathcal{D} \subseteq \mathcal{D}'$). This depends on e_p only through $e_p \bmod o_p$ (saturated) or exactly (unsaturated). The persistence test, “all $d \equiv 1 \pmod{b'}$ are $\equiv 1 \pmod{g}$ ”, is a statement about the same residue sets. Finally $c_p(e + \Delta)$ depends only on $c_p(e)$ and Δ . $\square$

Theorem 4.5 (Class closure and exact scaling). SOURCE RECORD: F.T51. *Scope C.*

(a) Closure principle. *If two states $n_1 < n_2$ of a persistent orbit have the same class, then with $P = n_2 - n_1$ and $\rho = L_{n_2}/L_{n_1} \in \mathbb{Z}_{\geq 1}$: for all $n \geq n_1$, $\gamma_{n+P} = \gamma_n$, the carries repeat, and*

$$x_{n+P} = \rho^2 x_n.$$

(b) Exact scaling. *Every persistent orbit is eventually class-periodic; hence eventually $x_{n+P} = \rho^2 x_n$ with $\rho \geq 2$.*

(c) Every orbit. *Every orbit of C eventually satisfies $x_{n+P} = \rho^2 x_n$ for some $P \geq 1$ and integer $\rho \geq 2$.*

Proof. (a) Induction on $n \geq n_1$ with Lemma 4.4: states n and $n+P$ have equal classes, so they have the same γ and the same carry vector, hence $L_{n+P}/L_n = \prod_p p^{(\text{sum of } \Delta_p \text{ over the period})} = \rho$ is constant, and $x_{n+P} = \gamma_{n+P} L_{n+P}^2 / g = \rho^2 x_n$.

(b) By Theorem 3.14, $\gamma_n \leq 2M$ eventually. A prime not yet in L can enter only through s' , i.e. only if $p^2 \mid w$ for some $w \leq 2M+1$: a finite set F of primes. After the last entry from F (or none), the class lies in a finite set; a deterministic map on a finite set is eventually periodic. $\rho = 1$ is impossible because $x_{n+1} > x_n$.

(c) $G_{n+1} \mid G_n$, so G_n is eventually constant, $= G_\infty$. If $G_\infty = 1$, Theorem 2.3 gives $x_{n+3} = 4x_n$ eventually. If $G_\infty > 1$ the orbit is eventually persistent and (b) applies. $\square$

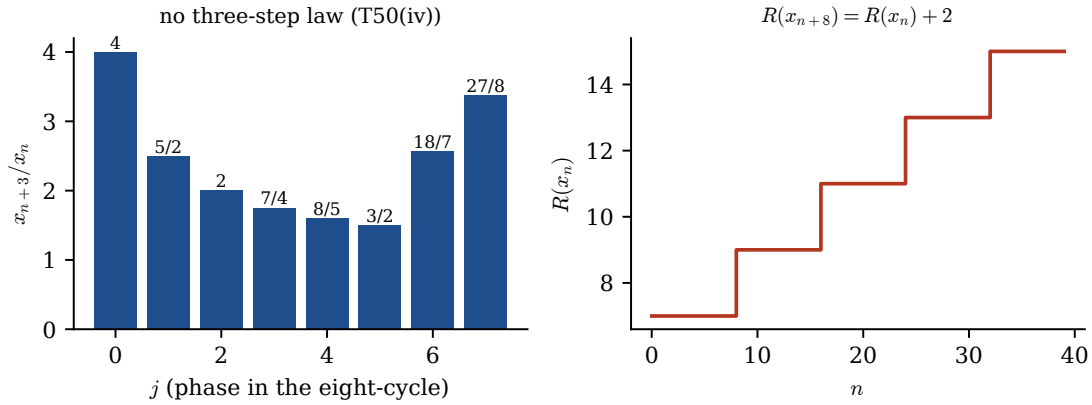


Figure 4.2: The eight-step orbit of Theorem 4.3, from exact iteration of C . Left: the eight values of x_{n+3}/x_n over one period; no single three-step law holds (Remark 4.1). Right: the fibre multiplicity $R(x_n)$, which rises by 2 every eight steps.

The class period may be far larger than the γ -period: the exponent of a carrying prime must run through o_p residues. For $q = 7$, $g = 367$ the class period is $2928 = 61 \cdot 48$ (Theorem 4.12). Because (a) is a proof of eternal periodicity once a class repeats, a finite computation on the residue machine is a complete certificate, not a sample.

4.4 Which carries occur

Theorem 4.6 (Window theorem). SOURCE RECORD: F.T52. *Scope C.* In a persistent orbit with modulus g :

- (a) if g is odd then $2 \mid L_n$ for all n ;
- (b) if g is odd then $g \nmid \gamma_n + 1$ for all n ; if $g \geq 4$ is even then $4g \nmid \gamma_n + 1$;
- (c) every complete carry (Theorem 4.8) has $M^2 < g$ for odd g and $M^2 < 4g$ for even g ; and $\gcd(M, g) = 1$.

Proof. (a) is Lemma 3.15. (b) If $g \mid w$ with g odd then $h = g$, $b' = 1$, and $x' = (w/g)L^2$ is an integer. By Theorem 4.2, $\mathcal{D}(x') = \{\text{all divisors } \geq 2 \text{ of } \sigma'\} \ni 2$ by (a), and $2 \not\equiv 1 \pmod{g}$, so x' is not persistent. For even g with $4g \mid w$: $v_2(w/g) \geq 2$, so $2 \mid \sigma'$, $b' = 1$, and again $2 \in \mathcal{D}(x')$, $2 \not\equiv 1 \pmod{g}$. (c) A complete carry starts from $\gamma = 1$ and passes through $w = 2, \dots, M^2$; by (b) the value $w = g$ (resp. $4g$) is forbidden, so $M^2 < g$ (resp. $< 4g$). $\gcd(M, g) = 1$ because $M \mid d \equiv 1 \pmod{g}$. $\square$

Consequence. A q -carry needs $g > q^2$ (g odd). There are only finitely many carry sizes for a given g (all $M < \sqrt{g}$); 5-carries do not exist for odd $g \leq 25$, 7-carries not for odd $g \leq 49$.

Theorem 4.7 (Absorption criterion). SOURCE RECORD: F.T53. *Scope C.* Let (g, E) be a persistent state and $w = \gamma + 1$. For $d \geq 2$ with $d \equiv 1 \pmod{g}$ put $j_p(d) = \max(0, v_p(d) - e_p)$ and $M(d) = \prod_p p^{j_p(d)}$ (the new part of d). Then d is available at the step iff $M(d)^2 \mid w$, and

$$L' = L \cdot r, \quad r = \text{lcm}\{M(d) : d \equiv 1 \pmod{g}, d \geq 2, M(d)^2 \mid w\}.$$

Proof. By Theorem 4.2, $d \mid \sigma'$ iff for all p , $j_p(d) \leq \lfloor v_p(w/h)/2 \rfloor$, iff $M(d)^2 \mid w/h$, iff $M(d)^2 \mid w$ (as $h \mid g$ is coprime to $M(d)$, since $\gcd(d, g) = 1$). Persistence supplies $d \equiv$

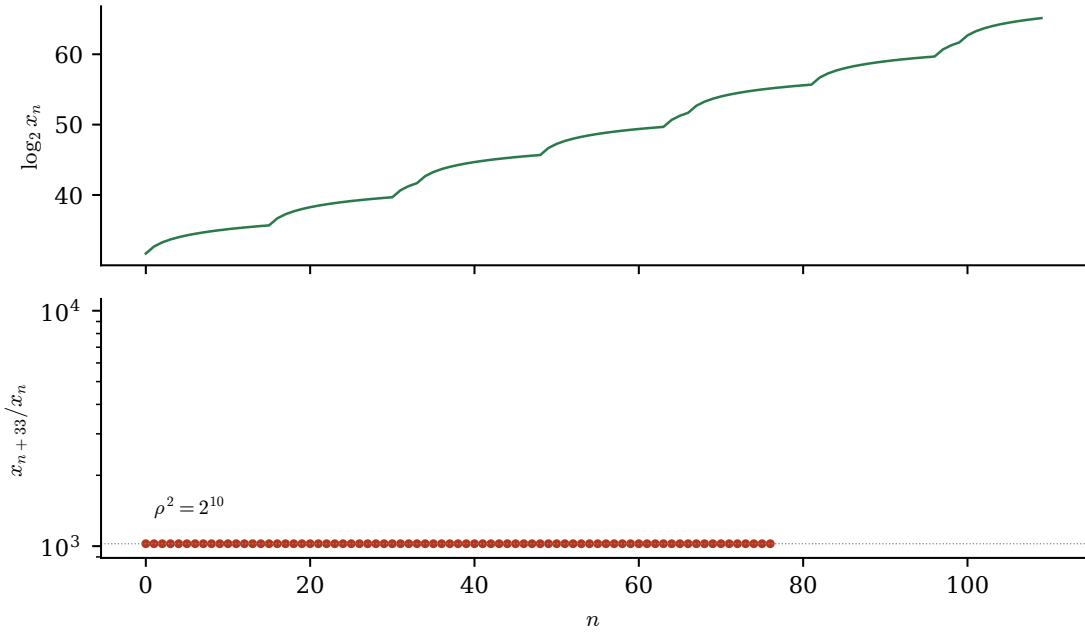


Figure 4.3: Exact asymptotic scaling for $g = 31$, single active prime 2. One class period has $P = 33$ steps and exponent increment 5, so $x_{n+33} = 2^{10}x_n = 1024x_n$ for all n in the eventual regime. Left: the orbit $\log_2 x_n$ with the period marked; right: the ratio x_{n+33}/x_n , exactly 1024 at every sampled n .

$1 \pmod{b'} \Rightarrow d \equiv 1 \pmod{g}$, so availability does not depend on h . The exponent of p in L' is $\max(e_p, \max_d v_p(d)) = e_p + \max_d j_p(d)$. $\square$

A step is of *no-absorption* type if no available d has $M(d) > 1$ ($r = 1$, $\gamma' = \gamma + 1$); *partial* if $r > 1$ and $r^2 < w$ ($\gamma' = w/r^2 \geq 2$); *complete* if $r^2 = w$ ($\gamma' = 1$). A complete carry has $h = 1$: indeed $r^2 \leq s'^2 \leq w/h$, so $w = r^2 \leq w/h$ forces $h = 1$.

Theorem 4.8 (Carry completeness). SOURCE RECORD: F.T54. *Scope C.* Let a carry occur at w from a plateau that started at γ_s (the value of γ right after the previous carry, or the initial value). Then

$$\gamma' \leq \gamma_s/4 + 1, \quad \text{and if } \gamma_s = 1: \quad w = M^2 = r^2 \text{ and } \gamma' = 1 \text{ (complete)}.$$

Hence in a persistent orbit γ_s strictly decreases at every carry as long as $\gamma_s \geq 2$; after finitely many carries $\gamma = 1$, and from then on every carry is complete, $w = M^2$, with cycle length $M^2 - 1$.

Proof. L is constant along the plateau, so by Theorem 4.7 availability of d depends only on $M(d)^2 \mid w$. Let d be available at the carry step. If $w - M(d)^2 \geq \gamma_s + 1$, then d was already available at the earlier plateau step $w'' = w - M(d)^2$ (since $M(d)^2 \mid w''$), i.e. the plateau would have ended there: a contradiction. So $w \leq \gamma_s + M(d)^2$. Since r is a multiple of $M(d)$ and $r \geq 2$,

$$\gamma' = \frac{w}{r^2} \leq \frac{\gamma_s + M(d)^2}{r^2} \leq \frac{\gamma_s}{4} + 1.$$

If $\gamma_s = 1$: $M^2 \mid w$ and $M^2 \leq w \leq 1 + M^2$ give $w = M^2$ (since $2M^2 > M^2 + 1$); then $r^2 \mid M^2$ and $r \geq M$ give $r = M$, $\gamma' = 1$. The last assertion: $\gamma_s \geq 2$ gives $\gamma' \leq \gamma_s/4 + 1 < \gamma_s$, and γ_s is a positive integer; plateaus are finite and carries infinite by Theorem 3.14. $\square$

So the cycle 1, 2, 3 of regime I is the case $M = 2$ of a general law: the γ -cycle is $1, \dots, M^2 - 1$ with M the minimal available carry.

Theorem 4.9 (Eventual carries are prime powers). SOURCE RECORD: F.T55. *Scope C. In the eventually periodic regime every carry M is a prime power p^j .*

Proof. A carry type that recurs infinitely often raises each of its primes' exponents infinitely often, so each prime $p \mid M$ has $e_p \geq o_p - 1$ (saturated) from some time on. Suppose M has two distinct primes and let d be available with $M(d) = M$ (so $w = M^2$, Theorem 4.8). Take a prime $q \mid M$ with $v_q(M) = j$. Choose $i \in [0, o_q - 1]$ with $i \equiv e_q + j \pmod{o_q}$; since $e_q \geq o_q - 1$, $i \leq e_q$. Then $d' = d q^{i-e_q-j}$ is an integer, $d' \equiv d \equiv 1 \pmod{g}$, $d' \geq 2$ (it still contains the other prime of M), and $M(d') = M/q^j < M$ with $M(d')^2 \mid M^2 = w$. So a smaller carry was already available at $w' = M(d')^2 < M^2$, contradicting that the plateau reached $w = M^2$. $\square$

Composite carries (for example $M = 6$, $w = 36$) occur only in the transient, while some participating prime is unsaturated.

Theorem 4.10 (Odometer law). SOURCE RECORD: F.T56. *Scope C. In the eventual regime let $\mathcal{P}$ be the set of active primes (unbounded exponent; all saturated) and call the others inert (constant exponent). For $p \in \mathcal{P}$ define*

$$\text{Abs}_p = \{k \bmod o_p : \exists a \in \mathcal{A}_{-p}, p^k a \equiv 1 \pmod{g}\},$$

where $\mathcal{A}_{-p}$ is the set of residues of divisors of $\prod_{q \neq p} q^{e_q}$ ($\langle q \rangle$ for active q ; a fixed finite set for inert q). $\text{Abs}_p \subseteq \mathbb{Z}/o_p$ is time-independent. At every state with $\gamma = 1$, the next carry is p^{j_p} for the active p minimising p^{j_p} , where

$$j_p = \min\{j \geq 1 : (e_p + j) \bmod o_p \in \text{Abs}_p\}.$$

For a single active prime p the carry exponents are the successive gaps of Abs_p around $\mathbb{Z}/o_p$, the γ -cycle lengths are $p^{2j} - 1$, and over one lap $P = \sum_i (p^{2j_i} - 1)$ and $\rho = p^{\sum_i j_i}$.

Proof. By Theorem 4.8 the next carry from $\gamma = 1$ is the smallest M available at $w = M^2$; by Theorem 4.9 it is a prime power; and p^j is available iff $p^{e_p+j}a \equiv 1$ for some $a \in \mathcal{A}_{-p}$ (no other prime has a new part). $\mathcal{A}_{-p}$ does not change in time, so Abs_p is fixed. $\square$

Example 4.11 ($g = 31$, $p = 2$). Here $o = 5$ and $\text{Abs} = \{0, 2, 4\}$, with gaps 2, 2, 1. The carries are $2^2, 2^2, 2^1$ ($w = 16, 16, 4$), so $P = 15 + 15 + 3 = 33$ and $\rho = 2^5 = 32$; thus $x_{n+33} = 1024x_n$ (Figure 4.3). For $g = 59$ the carries are mostly 3-carries ($w = 9$) with occasional 4-carries ($w = 16$). The wheel of Figure 4.4 shows Abs as marked positions on $\mathbb{Z}/o_p$.

There is no other kind of eventual cycle: every eventual cycle of a persistent orbit is a word in prime-power carries generated by independent odometers. Chapter 5 shows that in fact only one odometer runs.

4.5 The q -carry family

Theorem 4.12 (The q -carry theorem). SOURCE RECORD: F.T57. *Scope C. Let q be an odd prime and g an odd prime with $g > q^2$. Put $o = \text{ord}_g(q)$, $H = \langle q \rangle \leq (\mathbb{Z}/g)^\times$. Assume*

(Q1) $2 \notin H$; let $e_2 = \min\{e \geq 1 : 2^e \in H\}$ and $K = \langle 2 \rangle H$;

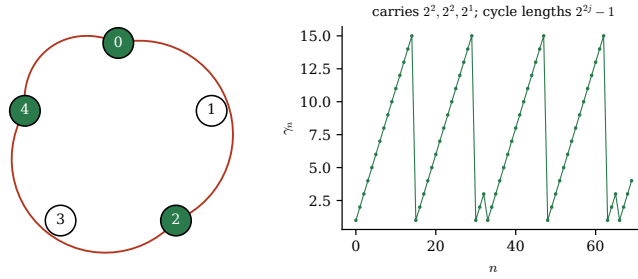


Figure 4.4: The odometer of Theorem 4.10 for $g = 31$, $p = 2$: the exponent $e_p \bmod 5$ moves around the wheel $\mathbb{Z}/5$; positions in $\text{Abs} = \{0, 2, 4\}$ are marked, and each carry advances the pointer to the next marked position. The gaps 2, 2, 1 are the exponents of the carries $2^2, 2^2, 2^1$.

(Q2) for every odd M with $3 \leq M \leq q - 1$, $M \bmod g \notin K$; for every even M with $2 \leq M \leq q - 1$, $M \bmod g \notin H$;

(Q3) u is a prime with $u \equiv q \pmod{g}$, $u > q$; seed exponents $E_0 = \{u : o - 1, q : \geq 1, 2 : e_2\}$;

(Q4) $G(x_0) = g$ for $x_0 = L_0^2/g$, $L_0 = u^{o-1}q^{e_q}2^{e_2}$ (an exact, finite divisor check).

Then the orbit of x_0 is persistent forever and

- (i) $\gamma_n = (n \bmod (q^2 - 1)) + 1$ and $L_{n+q^2-1} = qL_n$; hence $x_{n+q^2-1} = q^2x_n$;
- (ii) every carry is the q -carry at $w = q^2$, and every other step has $r = 1$;
- (iii) $R(x_n) = 2e_q(n) + 1$: it grows by 2 per carry.

Proof. Induction on the carries. State: γ and exponents $(e_u, e_q, e_2) = (o - 1, e, e_2)$, $e \geq 1$. All $w \leq q^2$ are $< g$, so $h = 1$, $b' = g$, and persistence is automatic from Theorem 4.2 once all $d \equiv 1 \pmod{g}$. The residues of the u -part and q -part together make up H ($u^c \equiv q^c$, $c \in [0, o - 1]$), independent of e . The only other prime is 2 with exponent $e_2 \geq 1$ (no other prime has $p^2 \mid w$, since the primes below q have exponent 0 and $u > q$).

Available new parts at $w < q^2$. A new part $M(d) = M$ has $M^2 \mid w < q^2$, so $M \leq q - 1$, a product of primes $< q$. If M is odd, 2 is not in M , the 2-part ranges over $2^0, \dots, 2^{e_2}$ (which represent all of $\langle 2 \rangle$ modulo H), and $d \equiv 1$ needs $M \in \langle 2 \rangle H = K$: excluded by (Q2). If M is even, the 2-exponent is exactly $e_2 + j$ with $2^{e_2} \in H$, and $d \equiv 1$ needs $M \in H$: excluded by (Q2) (and $2 \notin H$ by (Q1) covers $M = 2$). So $r = 1$ at every $w \leq q^2 - 1$.

At $w = q^2$. $\sigma' = qL$; the new part $M = q$ is available iff $q^{e+1}u^c(2\text{-part with } 2^a \in H) \equiv 1$ for some $c \leq o - 1$, i.e. $q^{e+1+c} \equiv 1$: $c \equiv -(e + 1) \pmod{o}$ works. No other new part has $M^2 \mid q^2$. So $r = q$, $\gamma' = q^2/q^2 = 1$. The state after the carry has the same shape with $e \rightarrow e + 1$, proving (i) and (ii). For (iii): $d = 2^a q^b u^c \equiv 1$ forces $2^a \in H$, i.e. $a \in \{0, e_2\}$ ($2 \notin H$ and e_2 is the first return); for each (a, b) there is exactly one $c \bmod o$; so $R + 1 = 2(e + 1)$. $\square$

Instances (verified by direct iteration of C and by a class-closure certificate, Theorem 4.5): $q = 3$: $g = 11$ (seed $\{47 : 4, 3 : 1, 2 : 2\}$; the orbit of Theorem 4.3 is a different saturator, $5 \equiv 3^3$), $P = 8$, $\rho = 3$; $q = 5$: $g = 31$, $P = 24$, $\rho = 5$; $q = 7$: $g = 367$, seed $\{3677 : 60, 7 : 1, 2 : 3\}$, $P = 48$, $\rho = 7$, class period 2928, $G = 367$ exact. The smallest admissible g for $q = 3, 5, 7$ is 11, 31, 367.

Sharpness of (Q2). It is necessary in the template: for $g = 193$, $q = 7$, the orbit is a 3-carry instead, because $3 \in \langle 2 \rangle H$ although $3 \notin H$.

Example 4.13 (An even persistent value). For $g = 68$ there is a persistent orbit, started at the reset state $(g, \gamma, E) = (68, 1, \{3 : 3, 13 : 1, 3947 : 8, 281 : 4\})$; verified by direct iteration

of C and by class closure (Theorem 4.5). So persistence is not restricted to odd g : for even g the lcm L is odd, the collapse modulus is $4g$, and the same carry calculus (Theorems 4.2–4.10) applies with $M^2 < 4g$.

Remark 4.14 (Scope of the earlier uniform statements). SOURCE RECORD: F.T58. Scope C. The statements that every orbit is eventually $x_{n+3} = 4x_n$, or eventually of the form s^2t/g with $t \in \{1, 2, 3\}$, are true for the regimes of Chapter 2 and Theorems 3.2–3.5 and are not universal (Theorem 4.3; the t -coordinate runs through $1, \dots, q^2 - 1$ in Theorem 4.12). The precise identity is, for G unchanged over three steps and $x = \gamma L^2/g$,

$$x_{n+3} = 4x_n \iff \gamma_{n+3}L_{n+3}^2 = 4\gamma_nL_n^2 \iff \gamma_{n+3} = 4\gamma_n/\rho^2, \quad \rho = L_{n+3}/L_n \in \mathbb{Z}_{\geq 1}.$$

The pairs $(\rho, \gamma_{n+3}) = (2, \gamma_n)$ and $(1, 4\gamma_n)$ are sufficient; other pairs with $\rho^2 \mid 4\gamma_n$ are also allowed. The γ -cycle is $1, \dots, M^2 - 1$ with M the minimal available prime-power carry (Theorem 4.8); $M = 2$ gives $1, 2, 3$. Theorem 3.14 is consistent with all of this ($\gamma \leq 8 \leq 2M$ for the $g = 11$ orbit).

Chapter 5 Carry Scheduling and Universal Single-Prime Rigidity

Theorem 4.10 describes a persistent orbit in its eventual regime as a family of independent counters, one per active prime. This chapter proves that the family has exactly one member. Whatever the modulus g and whatever the seed, a single prime carries forever and every other prime is frozen. This is the rigidity of the volume's title, and it has three consequences: the scaling factor is a prime power (Theorem 5.6), the reach counting function is logarithmic for every orbit (Corollary 5.7), and the two natural clocks of two saturated primes have coprime periods (Theorem 5.8).

5.1 The carry scheduler

Notation as in Chapter 4: a persistent orbit with $G = g$, exponent state $E = (e_p)$, $o_p = \text{ord}_g(p)$, $w = \gamma + 1$. By Theorem 4.5 the orbit is eventually class-periodic. A prime is *active* if its exponent is unbounded and *inert* otherwise. After a finite time only active primes change their exponent, every active prime is saturated ($e_p \geq o_p - 1$), and (Theorems 4.8, 4.9) every carry is complete and a prime power p^j , with $\gamma = 1$ immediately before it and $w = p^{2j}$.

For a state E and a prime $p \nmid g$ put $e = e_p$ (0 if $p \nmid L$) and define the *pending carry*

$$j_p(E) = \min\{j \geq 1 : p^{e+j}a \equiv 1 \pmod{g} \text{ for some } a \in \mathcal{A}_{-p}(E)\}, \quad c_p(E) = p^{j_p(E)},$$

where $\mathcal{A}_{-p}(E)$ is the set of residues of divisors of $\prod_{q \neq p} q^{e_q}$ ($j_p = \infty$ if no $j \leq o_p$ works).

Theorem 5.1 (The carry scheduler). SOURCE RECORD: F.T59. Scope C. *At a reset state ($\gamma = 1$) of the eventual regime the next carry is carried out by the prime p with the smallest pending carry $c_p(E)$; there are no ties. Its exponent increases by j_p , no other exponent changes, the γ -cycle is $1, 2, \dots, c_p^2 - 1$, and the next reset occurs after $c_p^2 - 1$ steps. The orbit stays persistent iff $c_p^2 < g$ (g odd; $c_p^2 < 4g$ for g even). Moreover:*

- (a) *for an active p , c_p is a function of $e_p \bmod o_p$ alone: $c_p = p^{\text{gap}_p(e_p)}$ with $\text{gap}_p(e) = \min\{j \geq 1 : (e + j) \bmod o_p \in \text{Abs}_p\}$;*

- (b) for an inert prime c_p is constant;
- (c) after p fires, $e_p \bmod o_p \in \text{Abs}_p$, and each further firing moves $e_p \bmod o_p$ to the next element of Abs_p in the cyclic order of $\mathbb{Z}/o_p$, so p visits every element of Abs_p cyclically.

Proof. By Theorem 4.8 the next carry from $\gamma = 1$ is the smallest M available at $w = M^2$. By Theorem 4.7 a new part $M(d)$ is available iff $M(d)^2 \mid w$. In the eventual regime a composite $M(d)$ involves ≥ 2 primes whose exponents change, hence active, hence saturated; by the argument of Theorem 4.9 such an M is not minimal. So the minimal available M is a prime power p^j , and p^j is available iff $p^{e+j}a \equiv 1$ for some $a \in \mathcal{A}_{-p}$, i.e. iff $j \geq j_p$; the smallest available M is $\min_p p^{j_p}$. Ties are impossible: $p^a = q^b$ with $a, b \geq 1$ has no solution for $p \neq q$. The persistence window is Theorem 4.6(c). For (a): for a saturated prime, the residue set of $\langle q \rangle$ does not depend on e_q , and for an inert prime it is constant; so c_p depends on E only through e_p and the residues of the other primes, which are constant in time. Hence c_p depends on $e_p \bmod o_p$ alone. (b) is the same observation. (c): after p fires, $p^{e_p}a \equiv 1$ for some a , so $e_p \bmod o_p \in \text{Abs}_p$; the next firing advances to the next element by definition of gap_p . $\square$

5.2 Single-prime dominance

Theorem 5.2 (Single-prime dominance). SOURCE RECORD: F.T60. *Scope C. In the eventual regime of every persistent orbit (every g , every seed) exactly one prime is active. Equivalently: there are no x_0 and no primes $p \neq q$ such that p and q both carry infinitely often. The same holds for every abstract system of cyclic odometers $(\mathbb{Z}/s_i, C_i, p_i)$, $s_i \geq 1$, $C_i \neq \emptyset$, with the minimum-pending-carry rule.*

Proof. At least one prime is active. Otherwise L is eventually constant, and $\gamma_n \leq 2M$ eventually (Theorem 3.14) would make x_n bounded, while $x_{n+1} > x_n$.

At most one. For an active prime r let

$$m_r = \max\{r^{\text{gap}_r(k)} : k \in \text{Abs}_r\}$$

be its *worst-case pending carry*. Since every $k \bmod o_r$ lies in a gap of Abs_r , we have $c_r \leq m_r$ at all times after r has fired once (Theorem 5.1(a)). Suppose two distinct primes are active. Distinct primes never have equal powers, so the numbers m_r (r active) are pairwise distinct; let p be the active prime with the largest m_r , and q any other active prime, so $m_q < m_p$.

Choose $k^* \in \text{Abs}_p$ with $p^{\text{gap}_p(k^*)} = m_p$. Since p fires infinitely often and its residue runs cyclically through all of Abs_p (Theorem 5.1(c)), there is a reset time t , after both p and q have fired, at which $e_p \bmod o_p = k^*$, so $c_p = m_p$. From then on the exponent of p changes only when p fires, and p fires only when $c_p < c_r$ for every other prime r (Theorem 5.1). But $c_q \leq m_q < m_p = c_p$ at every later reset. So p never fires after t , contradicting that p is active.

The argument used only parts (a)–(c) of Theorem 5.1, so it holds verbatim for abstract odometer systems. $\square$

Corollary 5.3 (Starvation rule). *The unique active prime r satisfies $m_r < c_p$ for every inert prime p with a finite pending carry (apply the proof to the worst-case state of r : it fires from there only if $m_r < c_p$, and c_p is frozen). In particular, among saturated primes with nonempty Abs the survivor is the one with the smallest worst-case pending carry: the prime with the largest worst-case carry always starves.*

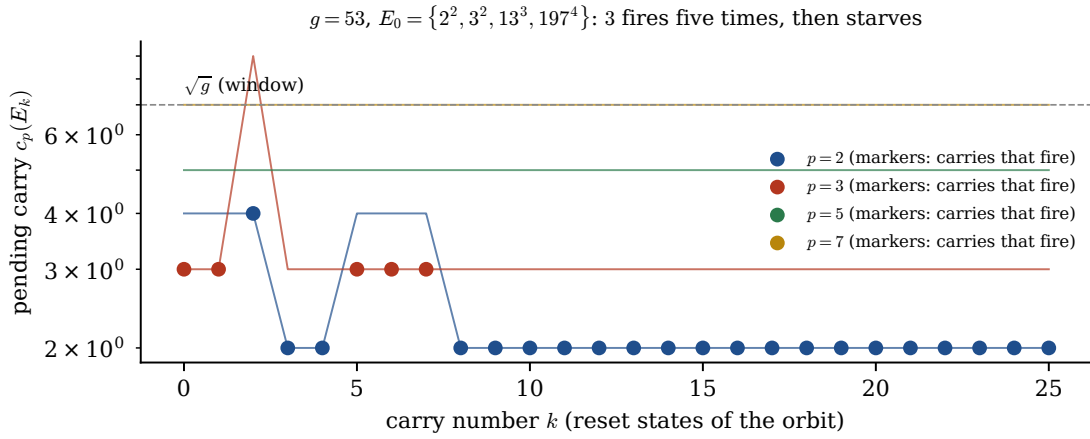


Figure 5.1: Prime-carry competition in an actual orbit at $g = 53$ (seed exponents $\{2:2, 3:2, 13:3, 197:4\}$, first sixteen steps cross-checked against direct iteration of C). The scheduler (Theorem 5.1) lets the prime with the smallest pending carry fire: the prime 3 fires five times during the transient, interleaved with 2, and then starves, after which the prime 2 carries alone, exactly as Theorem 5.2 and Corollary 5.3 require.

Corollary 5.4 (One rule for every regime). *Regime I ($G_\infty = 1, x_{n+3} = 4x_n$) is the single-prime system $p = 2$ with γ -cycle 1, 2, 3; the eight-step ($p = 3$), 24-step ($p = 5$), 48-step ($p = 7$) and mixed-power cycles are single-prime systems with other (p, Abs_p) .*

Corollary 5.5 (Threshold overlap is vacuous). *For two primes to coexist one would need $m_q > m_p$ and $m_p > m_q$. Hence the weaker necessary condition “ $B_p^+ > B_q^-$ and $B_q^+ > B_p^-$ ” on the extreme values of the candidate sets is true but never sufficient: it is never met by a surviving pair.*

5.3 Consequences

Theorem 5.6 (The scaling factor is a prime power). SOURCE RECORD: F.T61. *Scope C. Let p be the active prime (Theorem 5.2), s the minimal translation period of $\text{Abs}_p \subseteq \mathbb{Z}/o_p$ (the least $s \mid o_p$ with $\text{Abs}_p + s = \text{Abs}_p$), and $j_1, \dots, j_m$ the gaps of Abs_p inside one period, $\sum_i j_i = s$. Then the eventual orbit has γ -period*

$$P_\gamma = \sum_{i=1}^m (p^{2j_i} - 1), \quad L_{n+P_\gamma} = p^s L_n, \quad x_{n+P_\gamma} = p^{2s} x_n.$$

So $\rho = p^s$ is a prime power. For $G_\infty = 1$: $p = 2, s = 1, P_\gamma = 3, \rho = 2$. Every orbit of the collective map therefore eventually satisfies $x_{n+P} = p^{2s} x_n$ with p prime.

Proof. One active prime: the carries are the successive gaps of Abs_p around $\mathbb{Z}/o_p$ (Theorem 4.10), with cycle lengths $p^{2j} - 1$ (Theorem 4.8). The carry word is periodic with minimal rotation period equal to the minimal translation period s of Abs_p (its gap sequence repeats exactly when Abs_p is invariant under the translation), and over that period e_p increases by $\sum_i j_i = s$ while all other exponents are constant. The γ -sequence is determined by the carry word, so it is periodic with period P_γ , and $L_{n+P_\gamma} = p^s L_n$; then $x_{n+P_\gamma} = \gamma L_{n+P_\gamma}^2 / g = p^{2s} x_n$. $\square$

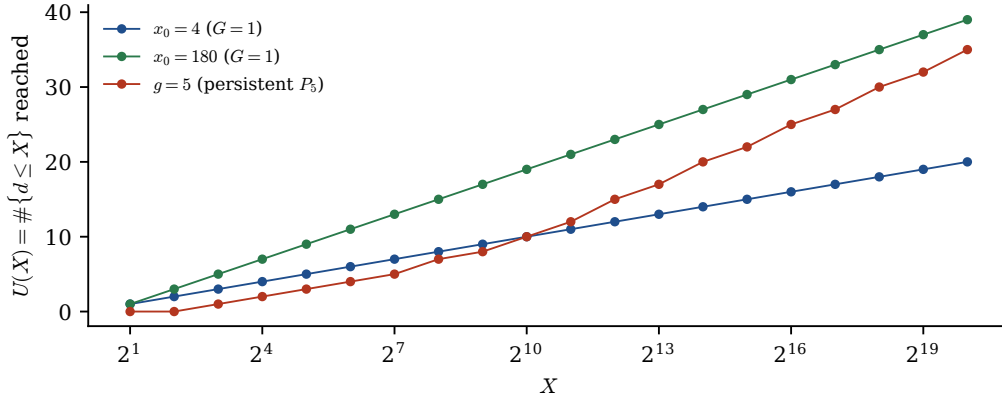


Figure 5.2: Reach counting $U(X) = \#\{d \leq X\}$ for three orbits (75 exact steps each): two orbits at $G = 1$ ($x_0 = 4$ and $x_0 = 180$) and the persistent orbit at $g = 5$ of Theorem 3.4. The growth is logarithmic in X , as Corollary 5.7 asserts, even though the fibre multiplicity $R(x_n)$ is unbounded.

(The class period of Theorem 4.5 may be a multiple o_p/s of P_γ , because the class records $e_p \bmod o_p$ while the carry word only sees it through Abs_p .) Examples: $g = 31$: $\text{Abs} = \{0, 2, 4\} \bmod 5$, $s = 5$, $\rho = 32$, $P_\gamma = 15 + 15 + 3 = 33$; $g = 17$: $s = 8$, $\rho = 256$; $g = 11, 31, 367$ with $\text{Abs} = \mathbb{Z}/o_p$: $s = 1$, $\rho = q$.

Corollary 5.7 (Logarithmic reach for every orbit). *For every orbit of C , the set U of all moduli d that ever occur in a fibre satisfies $U(X) = \#\{d \in U : d \leq X\} = O(\log X)$, with a constant depending on the orbit.*

Proof. If $G_\infty = 1$, Theorem 2.5 gives it. Otherwise the orbit is eventually persistent and, by Theorem 5.2, from some time n_0 on $L_n = cp^{e_n}$ with c constant, p the active prime and $e_n \rightarrow \infty$. By monotonicity (Theorem 1.3), every modulus ever occurring is either in the finite set $\bigcup_{n \leq n_0} \mathcal{D}(x_n)$ or divides cp^k for some k ; the second kind has the form $d = ap^i$ with $a \mid c$, and $d \leq X$ forces $i \leq \log_p X$. So $U(X) \leq |U_0| + \tau(c)(1 + \log_2 X)$. $\square$

The *unbounded multiplicity* of Chapter 1 ($R(x_n) \rightarrow \infty$) therefore coexists with *extreme sparsity* of the moduli that generate it: each orbit reaches only logarithmically many indices, all of the shape ap^i .

Theorem 5.8 (Coprime clocks). **SOURCE RECORD:** F.T62. *Scope C.* Let g be an odd prime and $p \neq q$ saturated primes with orders o_p, o_q and background F (the residues of the frozen part). Then $\mathcal{A}_{-p} = \langle q \rangle F$ is a union of cosets of $\langle q \rangle$ and $\text{Abs}_p = \{k : p^{-k} \in \langle q \rangle F\}$ is invariant under $k \mapsto k + \tau_p$, $\tau_p = o_p / \gcd(o_p, o_q)$; likewise Abs_q is τ_q -periodic with $\tau_q = o_q / \gcd(o_p, o_q)$. The numbers τ_p and τ_q are coprime.

Proof. $(\mathbb{Z}/g)^\times$ is cyclic, so $|\langle p \rangle \cap \langle q \rangle| = \gcd(o_p, o_q)$, and $p^\tau \in \langle q \rangle$ iff τ is a multiple of $o_p / \gcd(o_p, o_q)$. Then $p^{-(k+\tau_p)} \in \langle q \rangle F \iff p^{-k} \in \langle q \rangle F$. The numbers $o_p / \gcd(o_p, o_q)$ and $o_q / \gcd(o_p, o_q)$ are coprime. $\square$

Two clocks can therefore be coupled only through the common background F , and their natural periods share no factor, so phase locking is excluded by the arithmetic as well; Theorem 5.2 shows that the absence of coupling is not even needed: the scheduler rule alone excludes coexistence.

Remark 5.9 (Scope of the coexistence question). SOURCE RECORD: F.T63. Scope C. The “threshold-overlap” lemma (if p, q both occur infinitely often then $B_p^+ > B_q^-$ and $B_q^+ > B_p^-$) is true and is sharpened by Theorem 5.2 to the contradictory pair $m_q > m_p$, $m_p > m_q$. In particular no pair of counter-rotating odometers with alternating carry thresholds exists, even in the abstract system: with the minimum-rule scheduler the odometer with the larger worst-case pending carry is always the one that stops. Every candidate pair of coexisting primes is excluded by proof, not by search. The proof of Theorem 5.2 does not use primality of g , so the statement covers even and composite moduli as well; the numerical sample behind the examples is almost entirely prime g .

Chapter 6 Predictive Selection and Arithmetic Certification

Chapter 5 proved that one prime survives. This chapter answers the next question: *which one*, from finite arithmetic data. Three results do this. The carry-level predictor (Theorem 6.2) computes the next carry exactly; the minimax theorem (Theorem 6.3) shows that, in a system of independent odometers, the survivor is the prime with the smallest worst-case carry whatever the starting phases; and the static certificate (Theorem 6.5) turns this into a finite residue test at a single reset state, which certifies the whole infinite future.

Remark 6.1 (Hypotheses of certification). Static certification is proved for *prime* g , or more generally for g with no prime factor $\leq m^2$ (hypothesis (S5) below). It does not extend to composite or even g : for the persistent values $g = 15, 39, 51, 57, 68, 87$, every reset state fails (S5) (Theorem 6.8). The theorems below state this hypothesis where it is used.

6.1 The exact carry-level predictor

A *reset state* is a state with $\gamma = 1$. For $M \geq 2$ and a state E , $M = \prod p^{j_p}$ is *available* if there is $a \in \prod_{q \nmid M} \{q^i : 0 \leq i \leq e_q\}$ with $\prod_{p \mid M} p^{e_p + j_p} \cdot a \equiv 1 \pmod{g}$.

Theorem 6.2 (Exact carry-level predictor). SOURCE RECORD: F.T67. Scope C. *From a reset state E (exact $G = g$) of a persistent orbit, the next carry is*

$$M^* = \min\{M \geq 2 : M \text{ available at } E\}.$$

The next state is $E' = E$ with e_p increased by $v_p(M^*)$ for every $p \mid M^*$; $\gamma' = 1$ again, and the carry occupies the γ -steps $1, 2, \dots, M^{*2} - 1$ ($w = 2, \dots, M^{*2}$). The orbit remains persistent through this carry iff $M^{*2} < g$ when g is an odd prime (in general iff in addition $\gcd(g, w) = 1$ for all $w \leq M^{*2}$, e.g. when every prime factor of g exceeds M^{*2}). Consequently every state after the first reset state is a reset state, and the orbit is the iteration $E \mapsto E'$ of this map: there is no partial absorption after the first reset.

Proof. By Theorem 4.7 a divisor $d \equiv 1 \pmod{g}$ is available at the step w iff $M(d)^2 \mid w$, with $M(d)$ the new part. Write $d = \prod_{p \mid M} p^{e_p + j_p} \cdot a$ with $M = M(d)$ and a built from primes $q \nmid M$ with $v_q(a) \leq e_q$: this is exactly the statement “ M is available”. The first $w \geq 2$ for which some d has $M(d) > 1$ and $M(d)^2 \mid w$ is $w = M^{*2}$, $M^* = \min M(d)$ (Theorem 4.8: from $\gamma = 1$ the plateau runs to $w = M^{*2}$). At $w = M^{*2}$ the available new parts are those $M(d)$ with $M(d) \mid M^*$; by minimality only $M(d) = M^*$, so $r = \text{lcm} = M^*$ and $\gamma' = w/r^2 = 1$. For prime

g , all $w \leq M^{*2}$ are $< g$ exactly when $M^{*2} < g$; then $h = 1$ and persistence is automatic by Theorem 4.2. $\square$

6.2 The minimax selection theorem

Setting (independent odometers). Primes $p_1 < \dots < p_k$, each with a nonempty set $\text{Abs}_i \subseteq \mathbb{Z}/o_i$ and a state $e_i \in \mathbb{Z}/o_i$. The pending carry is $c_i(e_i) = p_i^{\text{gap}_i(e_i)}$, $\text{gap}_i(e) = \min\{j \geq 1 : (e + j) \bmod o_i \in \text{Abs}_i\}$; the smallest pending carry fires and $e_i \leftarrow e_i + \text{gap}_i(e_i)$. Let d_i be the largest cyclic gap of Abs_i (differences of consecutive elements, with $o_i + a_1 - a_r$ for the wrap-around) and $m_i = p_i^{d_i}$, the *worst-case carry*.

Theorem 6.3 (Minimax selection). SOURCE RECORD: F.T64. *Scope C.*

- (i) $c_i(e) \leq m_i$ for every phase e .
- (ii) Every walk, from every start phase, ends in a cycle in which exactly one prime fires, namely $p_* = \arg \min_i m_i$.
- (iii) With a window (a pending carry $\geq W$ is forbidden, i.e. the orbit leaves persistence), the system stays admissible forever iff $m_{p_*} < W$; if $m_{p_*} \geq W$, every walk exits in finite time.

Proof. (i) The phase e lies in a cyclic interval between consecutive elements of Abs_i (or is one of them); $\text{gap}(e)$ is the distance to the right end, hence at most the length of that interval, hence $\leq d_i$. (ii) Some prime fires infinitely often (finitely many primes, one fires per step). Suppose r fires infinitely often and $r \neq p_*$. The phase of r takes every value of Abs_r cyclically, in particular the one before a gap of length d_r , where its pending carry is m_r . There r fires only if $m_r < c_{p_*}(e_{p_*}) \leq m_{p_*}$ by (i). But $m_r > m_{p_*}$ (the m_i of distinct primes are distinct, and p_* has the smallest). Contradiction; so only p_* fires infinitely often. (iii) If $m_{p_*} < W$, the smallest pending carry is $\leq c_{p_*} \leq m_{p_*} < W$ at every step, so there is no exit. If $m_{p_*} \geq W$, then p_* eventually sits in its worst-case phase, where the smallest pending carry is $m_{p_*} \geq W$ (every competitor's pending carry is frozen above it): exit. $\square$

This sharpens Theorem 5.2, which proved “at most one”, to a non-iterative, phase-independent selection rule. It does not by itself apply to the collective map, because there the odometer tables depend on the actual state (Theorem 6.5).

6.3 The quotient-residue formula

Theorem 6.4 (Quotient-residue formula). SOURCE RECORD: F.T65. *Scope C.* Let p be a prime with $\gcd(p, g) = 1$; $H \leq (\mathbb{Z}/g)^\times$ the subgroup generated by the residues of the saturated other primes ($e \geq o - 1$); F the set of residues of the divisors of the frozen (unsaturated or inert) other primes' part; t_p the order of pH in $(\mathbb{Z}/g)^\times/H$. Put

$$S_p = \{r \bmod t_p : p^r f \in H \text{ for some } f \in F\}.$$

Then

$$\text{Abs}_p = \{k \bmod o_p : k \bmod t_p \in S_p\},$$

a union of residue classes modulo t_p (with $t_p \mid o_p$); the gaps of Abs_p in $\mathbb{Z}/o_p$ are the gaps of S_p in $\mathbb{Z}/t_p$, so $m_p = p^{\max \text{gap}(S_p)}$ (cyclic maximum in $\mathbb{Z}/t_p$). For prime g , $t_p = o_p / \gcd(o_p, |H|)$.

Proof. $\text{Abs}_p = \{k : p^{-k} \in \mathcal{A}_{-p}\}$ with $\mathcal{A}_{-p} = H \cdot F$ (each saturated prime contributes a whole cyclic subgroup, and their product is H). Now $p^{-k} \in HF$ iff $p^{-k}H = fH$ for some $f \in F$, i.e. iff $p^k f \in H$. This depends only on $k \bmod t_p$ since $p^{t_p} \in H$. For prime g the unit group is cyclic, $|\langle p \rangle \cap H| = \gcd(o_p, |H|)$, so $t_p = o_p / |\langle p \rangle \cap H|$. $\square$

The formula replaces “walk the odometer” by a finite computation in a quotient group; it holds for composite g as well (there the quotient need not be cyclic and t_p is the plain order), and it generalises Theorem 5.8 to any saturated background.

6.4 The static certificate

Theorem 6.5 (Static survivor certificate). SOURCE RECORD: F.T66. *Scope C.* Hypotheses (finite data of one reset state E). g is prime, or more generally g has no prime factor $\leq m^2$ (S5); $G(x) = g$ exactly; E is a reset state ($\gamma = 1$); a prime $r \nmid g$ is saturated in E ($e_r \geq o_r - 1$). Let Abs_r be computed from the other primes’ residue sets (Theorems 4.10, 6.4), d_r its largest cyclic gap and $m = r^{d_r}$.

(S3) $m^2 < g$;

(S4) no M with $2 \leq M < m$ and $\gcd(M, r) = 1$ is available at E ;

(S5) no prime factor of g is $\leq m^2$.

Conclusion. Under (S3)–(S5) the orbit from E is persistent forever and only r carries: the carries are r^j with j running cyclically through the gaps of Abs_r (starting at the phase $e_r \bmod o_r$); the γ -period is $\sum_i (r^{2^{j_i}} - 1)$ over one period s of Abs_r , $\rho = r^s$, and $x_{n+P} = r^{2^s} x_n$ for all $n \geq 0$ (not only eventually). Conversely, for prime g : if r is saturated, E is a reset state, and the orbit carries only r from E on and stays persistent, then (S3) and (S4) hold ((S5) is automatic for prime $g > m^2$).

Proof. Sufficiency. Induction on carries. Let E be a reset state with (S3)–(S5) and r saturated, and let M^* be the smallest available M (Theorem 6.2).

(a) If $M = r^a s$ with $a \geq 1$, $s > 1$ coprime to r is available, then so is s alone: since r is saturated, choose $i \in [0, o_r - 1]$ with $i \equiv e_r + a \pmod{o_r}$; then $i \leq e_r$, and $d' = d r^{i - e_r - a}$ is a smaller divisor $\equiv d \equiv 1$ with new part s (the argument of Theorem 4.9). So the smallest available M is either a pure power r^a or coprime to r .

(b) By (S4) every coprime available M is $\geq m$, and $= m$ is impossible (m is a power of r); so every coprime available M exceeds m .

(c) A pure power r^a is available iff $(e_r + a) \bmod o_r \in \text{Abs}_r$ (Theorem 4.10); the smallest such a is the gap at the current phase, so $r^a \leq m$. Hence $M^* = r^{\text{gap}(\varphi)} \leq m < \text{every coprime available } M$, and $M^{*2} \leq m^2 < g$ by (S3), with $\gcd(g, w) = 1$ for all $w \leq M^{*2}$ by (S5). By Theorem 6.2 the carry is r^{gap} , and the state after it, E' , is E with e_r increased by the gap: again a reset state with r saturated and all other exponents unchanged. Abs_r and the availability of coprime M do not change: the residue set of r among the divisors is the full subgroup $\langle r \rangle$ (saturated) and every coprime M ignores the exponent of r . So (S3)–(S5) hold at E' and the induction continues, the phase moving along the gaps of Abs_r . The periodicity statement is Theorem 5.6.

Necessity (prime g). If r carries forever, it visits the phase at which its pending carry is m (equality in Theorem 6.3(i): the phase before the largest gap), where $M^* = m$. Any available coprime M would have to exceed m (it cannot equal a power of r), hence none is $< m$: (S4).

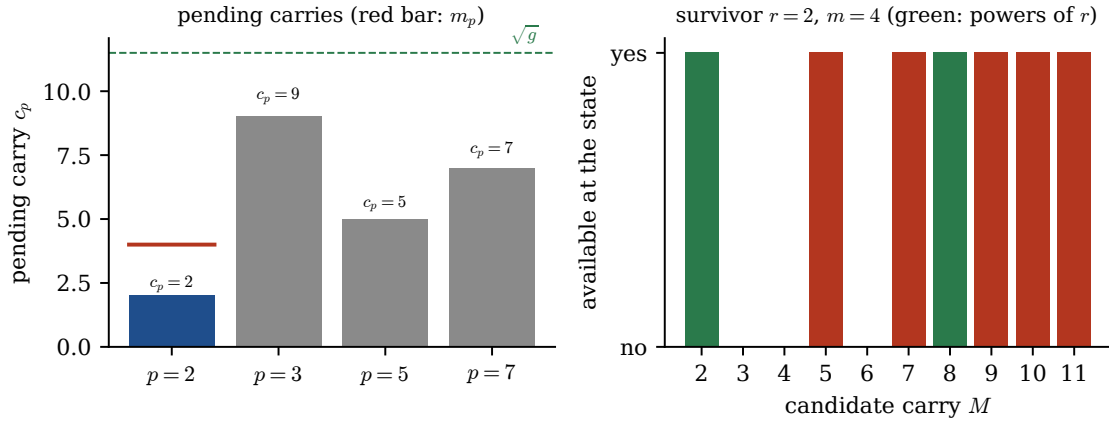


Figure 6.1: A statically certified reset state at $g = 127$ (Example 6.7). Left: pending carries c_p ; the red bar marks the worst-case carry m_p of the saturated prime. Right: availability of each candidate carry $M \leq \sqrt{g}$ at the state; green bars are the powers of the survivor $r = 2$. No $M < m = 4$ coprime to 2 is available, which is hypothesis (S4); $m^2 = 16 < 127$, which is (S3).

Persistence gives $m^2 < g$ (Theorem 4.6: the carry m passes through $w = g$ otherwise): (S3). For prime g , (S5) is then automatic. $\square$

Proposition 6.6 (Unsaturated extension and uniqueness). *Scope C. (Part of the source record F.T66, where it is stated as T66' and the uniqueness clause.) (a) If $e_r < o_r - 1$, apply Theorem 6.2 to the phases $e_r, e_r + \text{gap}, \dots$ until $e_r \geq o_r - 1$ (at most $o_r - 1$ carries; only r 's exponent moves as long as each next carry equals the predicted power of r), then Theorem 6.5: a finite check over r -carries only. (b) At most one prime is certified from a state.*

Proof. (a) is the concatenation of the two finite procedures. (b) If $r_1 \neq r_2$ were both certified, (S4) for r_1 gives $c_{r_2} > m_{r_1} \geq c_{r_1}$, and (S4) for r_2 gives $c_{r_1} > m_{r_2} \geq c_{r_2}$; so $c_{r_2} > c_{r_1} > c_{r_2}$, a contradiction. $\square$

What the certificate does not claim. The competitor condition (S4) concerns the *actual* state: the hypothetical tables of Theorem 6.3 for other primes are not used, because a competitor's Abs depends on r being saturated. Seeds that are not reset states ($\gamma_0 > 1$) and the moduli g discussed in Remark 6.1 lie outside the theorem; $\gamma_0 > 1$ is handled by the transient bounds of Chapter 7.

Example 6.7 (A certified state at $g = 127$). At the reset state $E = \{2:6, 3:6, 11:2, 41:6\}$ with $g = 127$ (checked by normalisation to have $\gamma = 1$), the pending carries are $c_2 = 4, c_3 = 9, c_5 = 5, c_7 = 7$; the prime 2 is saturated ($o_2 = 7$, Abs = $\{0, 2, 3, 4, 6\}$, largest gap 2, $m = 4$); $m^2 = 16 < 127$, and 127 is prime (S5); no coprime $M \in \{3\}$ with $M < 4$ is available. So (S3)–(S5) hold and the prime 2 carries forever, with P_γ given by Theorem 5.6. Figure 6.1 shows the pending carries against the worst-case bar m and the availability of each candidate M .

6.5 Consequences of certification

Theorem 6.8 (Consequences of static certification). SOURCE RECORD: F.T68. *Scope C.*

- (a) For prime g : every persistent orbit reaches a statically certified reset state (after its eventual regime is entered, Theorems 4.5, 5.2; every eventual reset state satisfies (S3)–(S5) for the active prime, by the necessity part of Theorem 6.5). So “which prime survives” is decided by a finite residue test at that state. For composite or even g this is not asserted: for $g = 15, 39, 51, 57, 68, 87$ every tested reset state fails (S5).
- (b) From a certified state the complete future is a function of finite data: the carry word (gaps of Abs_r from the current phase), the γ -period $\sum(r^{2^j} - 1)$, $\rho = r^s$, and the window margin $m^2 < g$.
- (c) g alone does not determine the survivor; the pair $(g, \text{background } E)$ does. For $g = 31$ there are orbits with survivor 2 and orbits with survivor 5.

Proof. (a) For prime g every eventual reset state has an active saturated prime r that carries forever from it and keeps the orbit persistent (Theorems 4.5, 5.2); the converse direction of Theorem 6.5 then gives (S3), (S4), and (S5) is automatic. (b) is the conclusion of Theorem 6.5. (c) For $g = 31$: the reset state $L_0 = 2^5 \cdot 101^2$ (so $E_0 = \{2:5, 101:2\}$) has survivor 2, with carries $2^2, 2^2, 2^1$, $P_\gamma = 33$ and $\rho = 32$ (Example 4.11); the template state of Theorem 4.12 with $q = 5$ has survivor 5, with $P = 24$, $\rho = 5$. Both are verified by direct iteration of C and by class closure (Theorem 4.5). $\square$

Remark 6.9 (Scope statement). The general phrasing “every persistent orbit reaches a statically certified reset state” is a theorem for prime g (Theorem 6.8(a)); for composite or even g the hypothesis (S5) fails at every reset state of the six cycles $g = 15, 39, 51, 57, 68, 87$, so no certified state is available there (the registered correction is Theorem 7.7), although each of these orbits still has exactly one active prime (Theorem 5.2) and an exact scaling law (Theorem 5.6). Closing this gap is Open Problem F.1.

Chapter 7 Effective Transient Bounds and Finite Predictability

Chapters 4–6 show that every orbit eventually enters a rigid, exactly scaling regime governed by one prime. This chapter makes the word *eventually* effective: from the finite data (g_0, γ_0) of the initial state alone, it bounds how long the transient can last. The bounds depend on neither the size of the exponents nor of the starting value. Five results do this: a sharp bound on carries until the first reset (Theorem 7.2), a class-counting bound on the entry into the final cycle (Theorem 7.3), a polynomial improvement (Theorem 7.4), a bound through changes of G (Theorem 7.6), and the registered scope of the whole programme (Theorem 7.7).

Definition 7.1 (Transient data). Let $f(a) = \lfloor a/4 \rfloor + 1$ and $\kappa(a) = \min\{k : f^k(a) = 1\}$ ($\kappa(1) = 0$). For g odd let $\mathcal{P}_g = \{p \nmid g : p^2 < g\}$, and for g even $\mathcal{P}_g = \{p \nmid g : p^2 < 4g\}$; by Theorem 4.6(c) these are the only primes that can carry after the first reset. Put

$$H(g) = \prod_{p \in \mathcal{P}_g} (2o_p - 1), \quad \Sigma = \sum_{p \in \mathcal{P}_g} o_p, \quad \omega = \max_{p \in \mathcal{P}_g} o_p, \quad \Phi = \sum_{p \in \mathcal{P}_g} (o_p - 1),$$

$$K_{\text{poly}}(g) = (\Phi + 1)(\Sigma + (\Sigma + 1)(\omega + 1) + 1).$$

A *carry event* is a step with $L' \neq L$; a *reducing event* is a step with $G' < G$ (Proposition 2.7); a *segment* is a maximal run of constant G . Carry events are counted, not collective-map steps; a plateau has at most $4g$ steps (Proposition 2.7).

κ	g	γ_0	carries	κ	g	γ_0	carries
1	61	2	1	7	151	10 531	7
2	73	5	2	8	19	42 663	8
3	61	37	3	9	11	173 739	9
4	73	155	4	10	79	333 226	10
5	193	345	5	11	103	1 345 610	11
6	31	1 387	6				

Table 7.1: Witnesses that Theorem 7.2 is sharp: in each case the state has $\gamma = \gamma_0$ with $\kappa(\gamma_0) = \kappa$ and the orbit needs exactly κ carries to reach $\gamma = 1$. The exponent states are listed in `witnesses.json` of the reproducibility package.

7.1 The reset-descent bound

Theorem 7.2 (Sharp reset-descent bound). SOURCE RECORD: F.T69. *Scope C.* In a segment of constant $G = g$ (any g , composite and even included) started at a state with $\gamma = \gamma_0$, the first reset state is reached after at most $\kappa(\gamma_0)$ carries, unless a reducing event ends the segment first. Moreover $\kappa(a) = \min\{k : 3a \leq 2 \cdot 4^k + 1\}$; equivalently $\kappa(a) = k$ iff $A_{k-1} < a \leq A_k$ with thresholds $A_k = (2 \cdot 4^k + 1)/3 = 1, 3, 11, 43, 171, 683, \dots$. Hence $\kappa(a) \leq \lceil \log_2 a \rceil$, and $\kappa(2) = 1$, $\kappa(10) = 2$, $\kappa(100) = 4$, $\kappa(1000) = 6$, $\kappa(10^6) = 11$. The bound is attained at the eleven levels $\kappa = 1, \dots, 11$: for each there are a prime g and a state with γ_0 whose orbit needs exactly $\kappa(\gamma_0)$ carries (Table 7.1). So the bound cannot be lowered at those levels; optimality at every other γ_0 is not asserted.

Proof. By Theorem 4.8, a carry from a plateau that started at $\gamma_s \geq 1$ gives $\gamma' \leq \gamma_s/4 + 1$; as γ' is an integer, $\gamma' \leq f(\gamma_s)$. The map f is nondecreasing, so after k carries $\gamma \leq f^k(\gamma_0)$ (at the plateau start), and the plateau start is 1 as soon as $f^k(\gamma_0) = 1$. Between carries γ only grows (Theorem 3.1(c)), so $\gamma = 1$ can only occur directly after a carry or at the start.

Closed form. Put $A_0 = 1$, $A_k = 4A_{k-1} - 1$, so $A_k = (2 \cdot 4^k + 1)/3$. Since $f(a) \leq A_{k-1} \iff \lfloor a/4 \rfloor \leq A_{k-1} - 1 \iff a \leq 4A_{k-1} - 1 = A_k$, induction gives $f^k(a) = 1 \iff a \leq A_k$ ($f \geq 1$, $f(1) = 1$). So $\kappa(a)$ is the least k with $a \leq A_k$, i.e. $3a \leq 2 \cdot 4^k + 1$. Finally $a \leq 2^k \Rightarrow a \leq A_k$ for $k \geq 1$ (as $A_k \geq 2^k$), giving $\kappa(a) \leq \lceil \log_2 a \rceil$.

Sharpness. Table 7.1 lists, for each $\kappa \leq 11$, a reset-reaching state whose carry count was obtained by exact iteration of C (for $\kappa \leq 8$) and by the residue machine, validated against direct iteration, for larger κ . $\square$

Scope. κ counts carries, not steps. A plateau has at most $4g$ steps, so the first reset is reached in at most $(\kappa(\gamma_0) + 1)4g$ collective steps.

7.2 The class-counting entry bound

Theorem 7.3 (Class-counting entry bound). SOURCE RECORD: F.T70. *Scope C.* Let a segment have constant $G = g$ and let s_0 be its first reset state.

- (a) If G stays g forever, the sequence of classes of the reset states $s_0, s_1, \dots$ repeats within $H(g)$ steps: the first repeated class occurs at index $j \leq H(g)$, the cycle is entered at an index $i \leq H(g) - 1$, and from s_i on the orbit is purely periodic ($x_{n+P} = \rho^2 x_n$, Theorem 4.5).

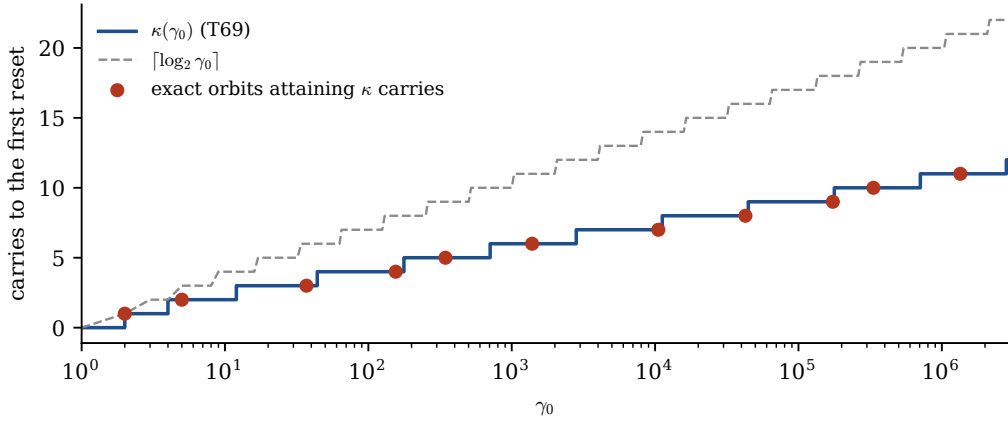


Figure 7.1: The reset-descent function $\kappa(a)$ (steps) against γ_0 on a logarithmic axis, with the witnesses of Table 7.1 marked at their γ_0 : each witness lies exactly on the staircase, so the bound of Theorem 7.2 is attained at each level $1 \leq \kappa \leq 11$.

- (b) If the segment ends with a reducing event, it ends before the carry number $H(g)$ after s_0 .
- (c) For prime g (more generally whenever Theorem 6.5 applies) the entry state s_i is statically certified, (S3)–(S5), for the unique active prime r .

So the number of carries from the start of the segment to a certified state is at most $\kappa(\gamma_0) + H(g) - 1$, independent of the exponents. The certification clause (c) is proved for prime g only; the bound (a)–(b) holds for every g .

Proof. After the first reset every carry is complete with $M^2 < g$ (for even g , $< 4g$) by Theorems 4.8, 4.6, so only primes of $\mathcal{P}_g$ change their exponent; all other exponents are constant. By Theorem 6.2 the next reset state is a function of the class vector $(c_p(e_p))_{p \in \mathcal{P}_g}$ (Lemma 4.4), and a class has at most $2o_p - 1$ values (exact below saturation, then o_p residues), so there are at most $H(g)$ class vectors.

(a) Among $s_0, \dots, s_H$ two share a class; the first repeat has index $j \leq H$, so the entry is at $i < j \leq H$, i.e. $i \leq H - 1$, and the closure principle (Theorem 4.5) makes the orbit periodic from s_i .

(b) If the segment lasted H carries after s_0 , the same pigeonhole argument would make it periodic, so no reducing event could occur (persistence of the next step is a function of the class, Lemma 4.4).

(c) In the cycle every carry recurs infinitely often, hence is a prime power (Theorem 4.9), and by Theorem 5.2 only one prime r carries. r is saturated: its exponent rises in every period while its class returns, which is impossible for an unsaturated class (exact). The orbit from s_i carries only r and stays persistent: this is the hypothesis of the necessity half of Theorem 6.5, so (S3), (S4) hold, and (S5) is automatic for prime g . The availability of coprime M does not depend on the phase of r (saturated), so (S4) holds at s_i as well as at the worst-case phase. $\square$

Values. $H(11) = 171$, $H(31) = 2\,655$, $H(59) = 21\,297\,195$; these are reproduced by the code of Appendix C.

7.3 The polynomial entry bound

Theorem 7.4 (Polynomial entry bound). SOURCE RECORD: F.T71. *Scope C. In the setting of Theorem 7.3 ($G = g$ from s_0 on; g prime, or every prime factor of g larger than the squares of all carries, so that Theorem 6.2 applies), the entry index and the exit index satisfy*

$$i \leq K_{\text{poly}}(g) = (\Phi + 1)(\Sigma + (\Sigma + 1)(\omega + 1) + 1),$$

a cubic polynomial in the multiplicative orders instead of the product $H(g)$. Values for $g = 11, 31, 59, 101, 197$:

$$\begin{aligned} K_{\text{poly}} &= 2\,688, 44\,928, 1\,243\,920, 10\,707\,144, 229\,237\,866, \\ H &= 171, 2\,655, 21\,297\,195, 386\,149\,351, 1.78 \cdot 10^{15}. \end{aligned}$$

So K_{poly} is the better bound from $g \approx 59$ on, and the minimum of the two is the bound. For $g = 211$: $H(211) \approx 2.4 \cdot 10^{13}$ and $K_{\text{poly}}(211) \approx 1.1 \cdot 10^8$.

Proof. Call $p \in \mathcal{P}_g$ *saturated* if $e_p \geq o_p - 1$; saturation is monotone (exponents never decrease).

L1 (reduction). The next carry M^* (Theorem 6.2) is either a pure power r^a of a saturated prime r , or its prime support lies entirely among unsaturated primes. Indeed, if $M = r^a s$ ($a \geq 1$, $s > 1$ coprime to r , r saturated) is available through d , then $d' = dr^{i-e_r-a}$ with $i \equiv e_r + a \pmod{o_r}$, $0 \leq i \leq e_r$, is a smaller divisor $\equiv 1 \pmod{g}$ with new part $s < M$ (the argument of Theorem 4.9), so M is not minimal.

L2 (frozen competitor). Availability of M with unsaturated support depends on the residues of the saturated primes only through the full subgroups $\langle g \rangle$, and on the exact exponents of the unsaturated primes. Hence $c_U := \min\{\text{available } M \text{ with unsaturated support}\}$ is constant as long as no unsaturated exponent changes.

Epochs. An *epoch* is a maximal run of carries that are pure powers of saturated primes; it ends with a carry of unsaturated support (an *epoch-ending* carry), or never. Such a carry raises $\Phi_{\text{cur}} = \sum_{p \in \mathcal{P}_g} \min(e_p, o_p - 1)$ by at least 1 (each prime of its support has $e_p < o_p - 1$), so there are at most Φ epoch-ending carries and at most $\Phi + 1$ epochs.

Inside an epoch the saturated primes are independent odometers $\mathbb{Z}/o_p$ with fixed tables Abs_p (Theorems 4.10, 6.4: the tables depend on the saturated set and on the frozen exact exponents), together with the frozen competitor c_U (if c_U ever wins, the epoch ends). Let $p^* = \arg \min m_p$ over the saturated primes (Theorem 6.3).

Claim A. Every saturated $r \neq p^*$ fires at most $|\text{Abs}_r| \leq o_r$ times in the epoch. After its first firing the phase of r lies in Abs_r , and each further firing moves it to the next element cyclically; within $|\text{Abs}_r|$ landings it lands on the element a^* before the largest gap, where its pending carry is m_r . There $c_{p^*} \leq m_{p^*} < m_r$ (Theorem 6.3(i)), so r never fires again.

Claim B. If p^* fires $|\text{Abs}_{p^*}| + 1$ times in a row (no other prime, no c_U in between), the system returns to the state it had after the first of these firings, hence repeats forever: the class cycle has been entered. So in an epoch that is not the final periodic one, runs of consecutive p^* -firings have length $\leq |\text{Abs}_{p^*}|$, and there are at most $N_o + 1$ such runs, where $N_o \leq \sum_{r \neq p^*} |\text{Abs}_r| \leq \Sigma$ is the number of non-survivor firings.

An epoch therefore contains at most $N_o + (N_o + 1)(|\text{Abs}_{p^*}| + 1) \leq \Sigma + (\Sigma + 1)(\omega + 1)$ saturated carries before it ends or its cycle begins, plus the ending carry. Summing over at most $\Phi + 1$ epochs gives K_{poly} . The exit index obeys the same bound, since the carry sequence is deterministic until the window is violated. $\square$

Remark 7.5 (Numerical erratum to the source data sentence). SOURCE RECORD: F.E71. Scope C. The data sentence of the source record quotes $H \approx 10^{15}$ and $K_{\text{poly}} \approx 2 \cdot 10^8$ for $g = 211$; the exact values are $H(211) = 24\,165\,141\,422\,031 \approx 2.4 \cdot 10^{13}$ and $K_{\text{poly}}(211) = 113\,903\,360 \approx 1.1 \cdot 10^8$ (the quoted figures match $g = 197$: $H(197) \approx 1.8 \cdot 10^{15}$, $K_{\text{poly}}(197) \approx 2.3 \cdot 10^8$). Only these two numerical expressions change; the theorem and its proof are unaffected. The values above were recomputed in exact integer arithmetic for this volume.

7.4 The dynamic- G effective bound

Theorem 7.6 (Dynamic- G effective bound). SOURCE RECORD: F.T72. *Scope C. Let x_0 be any state ($\mathcal{D}(x_0) \neq \emptyset$) with $G(x_0) = g_0$ and γ_0 . The orbit is a chain of segments $g_0 \supsetneq g_1 \supsetneq \cdots \supsetneq g_k \geq 1$ (each $g_{i+1} \mid g_i$), and:*

- (i) *the number of segments with $G > 1$ is at most $\Omega(g_0)$ (prime factors with multiplicity);*
- (ii) *at a reducing event $g_i \rightarrow g_{i+1}$: $\gamma_{i+1} = \frac{g_i+1}{g_i} \cdot \frac{\gamma_{\text{before}}+1}{r^2}$, $r = L'/L \geq 1$, and $\gamma_{\text{before}} \leq \gamma_i + 4g_i$; hence $\gamma_{i+1} \leq g_{i+1}(\gamma_i + 4g_i + 1)/g_i$, a function of the finite data only;*
- (iii) *segment i contains at most $\kappa(\gamma_i) + H(g_i) - 1$ carry events (before its reducing event, or before the final class cycle is entered);*
- (iv) *if the chain ends at $G = 1$ (regime I), the tail to the cycle $1 \rightarrow 2 \rightarrow 3$ takes at most $4\kappa_3(t)$ steps, where $\kappa_3(t) = \min\{k : f^k(t) \leq 3\}$, $t = \gamma$ at the start of the tail;*
- (v) *hence the total number of carry events before the orbit is in its final class cycle is at most $\mathcal{R}(g_0, \gamma_0)$, where $\mathcal{R}(1, \cdot) = 0$ and*

$$\mathcal{R}(g, \Gamma) = \kappa(\Gamma) + H(g) - 1 + \max_{g' \mid g, g' < g} \mathcal{R}(g', \lfloor g'(\Gamma + 4g + 1)/g \rfloor).$$

This is explicit and computable from (g_0, γ_0) alone.

Qualification on units. The principal bound (iii), (v) counts carry events, not iterations of C . The conversion to iterations uses three further estimates, each proved in this volume: a plateau has at most $4g_i$ steps (Proposition 2.7), so segment i takes at most $(\text{carries} + 1) \cdot 4g_i + 1$ steps; and the $G = 1$ tail takes at most $4\kappa_3(t)$ steps, (iv). The theorem gives an explicit bound; it does not give the exact transient time (Open Problem F.2), and it is dominated by $H(g_i)$ and astronomically larger than observed transients (Figure 7.2).

Proof. (i) $\mathcal{D}(x) \subseteq \mathcal{D}(x')$ and $G(x') \mid G(x)$ (Theorem 1.3); a strictly decreasing divisibility chain $g_0 > g_1 > \cdots > g_k \geq 2$ has $\Omega(g_0) \geq \Omega(g_k) + k \geq k + 1$.

(ii) $x' = (\gamma + 1)L^2/g$. At a reducing event $G' = g'$, $y' = g'x' = g'(\gamma + 1)L^2/g$, so $\gamma' = y'/L^2 = (g'/g)(\gamma + 1)(L/L')^2$ (Theorem 3.1(a); $L \mid L'$ because $\mathcal{D} \subseteq \mathcal{D}'$). Within a segment the plateau-start γ never rises ($f(a) \leq a$), and each plateau adds at most $4g$ to γ (Proposition 2.7), so $\gamma_{\text{before}} \leq \gamma_i + 4g_i$. With $r \geq 1$ this gives the stated inequality.

(iii) Theorem 7.2 gives the carries to the first reset and Theorem 7.3 those after it: the segment is a constant- G run, so Theorems 4.5, 4.6, 4.8, 6.2 apply to it verbatim, and since persistence of each step is a function of the class (Lemma 4.4), a segment that outlives H carries after its reset is periodic and never reduces. The step count follows from the plateau bound.

(iv) At $G = 1$ the state is $x = s^2t$, $\gamma = t$ squarefree (Theorems 2.2, 2.3), and $t \mapsto \text{sqf}(t + 1)$; the entry bound of Theorem 2.3 is $4\kappa_3(t)$.

(v) Combine (ii)–(iv) and maximise over the possible chains: the next segment starts with some $\gamma_{i+1} \leq \lfloor g_{i+1}(\gamma_i + 4g_i + 1)/g_i \rfloor$ and $\mathcal{R}(g', \cdot)$ is nondecreasing in its second argument. $\square$

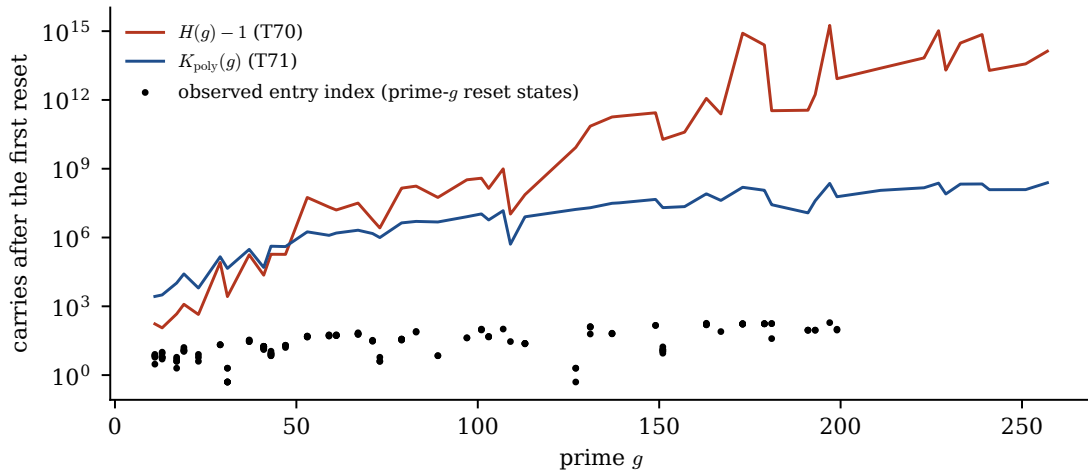


Figure 7.2: Observed versus certified transients. Each point is a random reset state at a prime $g < 200$, run exactly ($n = 150$ seeds); the vertical axis is the observed number of carries after the first reset before the cycle is entered. The curves are the bounds $H(g)$ (Theorem 7.3) and $K_{\text{poly}}(g)$ (Theorem 7.4). The largest observed index is 194 ($g = 197$); the bounds exceed 10^8 there. They are worst-case guarantees, not predictions, and the gap between the points and the curves is Open Problem F.2.

7.5 Registered scope of the programme

Theorem 7.7 (Registered corrections and scope). SOURCE RECORD: F.T73. *Scope C.*

- (E1) Certification scope. *Theorem 6.8(a) is proved for prime g . For composite or even g the hypothesis (S5) can fail while the orbit is persistent: in all six exact composite cycles tested ($g = 15, 39, 51, 57, 68, 87$) every reset state fails (S5). For such g the survivor exists (Theorem 5.2) and the class cycle is entered within $H(g)$ carries (Theorem 7.3(a)), but static certification as formulated in Theorem 6.5 is not available (Open Problem F.1).*
- (E2) Sharp constant. *The weaker form of the reset-descent bound, $\gamma' \leq \lfloor (\gamma + 1)/4 \rfloor \leq \lfloor \gamma/2 \rfloor$, loses a factor 2 per carry; the sharp statement is $\kappa(\gamma_0) \approx \log_4 \gamma_0$, and it is attained at the eleven levels of Table 7.1 (Theorem 7.2).*
- (E3) Index. *The entry index is $\leq H - 1$ (the first repeat is at $\leq H$); Theorem 7.4 supplies a bound that does not grow like a product over primes.*
- (E4) Effectiveness. *The question of a bound from an arbitrary seed before G stabilises is answered by Theorem 7.6 as an explicit bound; the exact transient time remains open.*

Proof. (E1) is Theorem 6.8 and the computation of Appendix C for the six cycles; (E2)–(E4) are the content of Theorems 7.2, 7.3, 7.4, 7.6. $\square$

7.6 Finite predictability

Combining the results of Chapters 4–7: for every initial state with $\mathcal{D}(x_0) \neq \emptyset$,

1. the orbit has a finite chain of values of G and eventually satisfies $x_{n+P} = \rho^2 x_n$ with $\rho = p^s$ a prime power (Theorems 4.5, 5.6);
2. exactly one prime carries in the final regime (Theorem 5.2), and for prime g it is decided by a finite residue test (Theorems 6.3–6.8);

3. the time to reach that regime is at most $\mathcal{R}(g_0, \gamma_0)$ carry events, computable from (g_0, γ_0) alone (Theorem 7.6).

In this sense the asymptotics of the collective dynamics are *effectively predictable*: existence (eventual exact scaling), structure (one prime, one carry word) and effectiveness (computable bounds on the transient) are all established, with the two scope qualifications stated in Theorems 7.6 and 6.5.

Appendix A Mathematical definitions and notation

This appendix collects the notation of the volume, with its correspondence to Volume I and to the symbols of the source records (FIELD-I). Throughout, $x \in \mathbb{Q}_{>0}$ is written $x = a/b$ in lowest terms.

A.1 Objects

Symbol	Meaning	Where defined
$\mathcal{D}(x)$	fibre: $\{d \geq 2 : d^2 \mid a, d \equiv 1 \pmod{b}\}$; the set of moduli d for which x is a rational nest value	Vol. I Thm 1.2; Def. 1.1
$R(x)$	fibre multiplicity $ \mathcal{D}(x) $	Chapter 1
$L(x)$	$\text{lcm } \mathcal{D}(x)$	Chapter 1
$G(x)$	$\gcd\{d - 1 : d \in \mathcal{D}(x)\}$	Chapter 1
$\Delta_{\text{coll}}(x)$	collective increment $L(x)^2/G(x)$, reduced with denominator G	Prop. 1.2
C	collective map $C(x) = x + L(x)^2/G(x)$	Def. 1.1
x_n	orbit $x_{n+1} = C(x_n)$	
$\gamma(x)$	$Gx/L^2 \in \mathbb{Z}_{>0}$; $x = \gamma L^2/g$	Prop. 1.2, Thm 3.1
$T(x), K_d$	growth time and its bound	Thm 1.5
s, t	$x = s^2 t$ at $G = 1$, t squarefree; sqf the squarefree part	Chapter 2
$\kappa_3(t)$	least k with $f^k(t) \leq 3$	Thm 2.3
$U, U(X)$	reach set and its counting function	Thm 2.5, Cor. 5.7
g, G_∞	value of G on a segment; eventual value	Chapter 3
P_g	persistent regime of Thm 3.2	Chapter 3
M	collapse modulus (g odd: g ; g even: $4g$)	Thm 3.14
$E = (e_p)$	exponent state $L = \prod p^{e_p}$	Chapter 4
o_p	$\text{ord}_g(p)$	
w, h, b', σ'	$w = \gamma + 1, h = \gcd(g, w), b' = g/h, \sigma' = L \prod p^{\lfloor v_p(w/h)/2 \rfloor}$	Thm 4.2
$c_p(e)$	class of an exponent	Chapter 4
$M(d)$	new part of a modulus d	Thm 4.7
r, ρ, P	growth ratio, scaling factor, period	Thm 4.5
Abs_p	absorption set $\subseteq \mathbb{Z}/o_p$	Thm 4.10
j_p, c_p, m_p	pending exponent, pending carry p^{j_p} , worst-case carry	Thms 5.1, 6.3
s (in Ch. 5)	minimal translation period of Abs_p	Thm 5.6
H, t_p	subgroup generated by saturated primes; order of pH	Thm 6.4
(S3)–(S5)	static certificate hypotheses	Thm 6.5
f, κ	$f(a) = \lfloor a/4 \rfloor + 1$; reset-descent function	Def. 7.1
$\mathcal{P}_g, H(g)$	prime set and class count	Def. 7.1
$\Sigma, \omega, \Phi, K_{\text{poly}}$	order sums and polynomial bound	Def. 7.1
$\mathcal{R}(g, \Gamma)$	dynamic- G bound	Thm 7.6

A.2 Scope tags

Each result carries a scope tag. **A0**: elementary occurrence arithmetic, independent of any choice of interaction rule. **C**: results about the collective map C of Definition 1.1, an explicitly selected rule; once C is selected, its dynamics are theorems.

A.3 Correspondence with Volume I and with the source records

- Volume I, Theorem 1.2 (fibre characterisation) gives $\mathcal{D}$; Theorem 1.7 (square-root reduction) is used in Lemma 2.1; Theorem 2.2 (pair future) and Theorem 2.5 (collective rendezvous and coherence) motivate G , L and the choice of C in Chapter 1.
- Source records use $D(x)$ for $\mathcal{D}(x)$, and $\text{lcm}(d^2)/G$ for the collective increment; the equality $\text{lcm}(d^2) = L^2$ is immediate from valuations (Chapter 1). The source records **F.T48**, **F.T58**, **F.T63** appear here as scope remarks (Remarks 4.1, 4.14, 5.9).
- The registry chapter field of a record (II.1–II.7) records the order of its original release; the manuscript chapter is given in Appendix B. For example, the γ -framework (**F.T44**) and the bounded window (**F.T47**) have registry chapter II.4 and appear in Chapter 3, where they are first used.

Appendix B Theorem registry and proof dependencies

The source registry contains 128 records across the coordinated volumes; 68 are assigned to this volume. The crosswalk below identifies them by their original registry IDs (the prefix **F.** is omitted; for example **T51** stands for **F.T51**). It lists the statement kind recorded in the registry, the registered scope tag, the registry chapter (the order in which a result was first released) and the location in this manuscript. The two chapter schemes differ for a few records, because results are placed here where they are first needed: **T31** (monotone orbits) is in Chapter 1, and **T44**, **T47** (the γ -framework and the bounded window) are in Chapter 3. Registered corrections and open problems are included, so that every record can be traced.

ID	Source record title	Source status	Scope	Reg. chap.	Here
CDEF	G1-C collective map C	PROPOSED	C	II.1	Def. 1.1
T17p	Delta_coll is reduced	PROVED+CHECKED	A0	II.1	Prop. 1.2
T17	Plateau law	PROVED+CHECKED	C	II.1	Prop. 1.4
T18	Growth-time bound	PROVED+CHECKED	C	II.1	Thm 1.5
T19	Collective fibre growth	PROVED	C	II.1	Thm 1.8
T26	Exact global minimum of the growth-time bound	PROVED+CHECKED	C	II.1	Prop. 1.6
T27	Provable lower bound for growth of R	PROVED+CHECKED	C	II.1	Prop. 1.9
T28	Monotone fibre growth is specific to the collective branch	PROVED+CHECKED	C	II.1	Ex. 1.10
T30	Arithmetic of G	PROVED	A0	II.2	Thm 2.2
T31	Monotone orbits	PROVED	C	II.2	Thm 1.3

ID	Source record title	Source status	Scope	Reg. chap.	Here
T32	Regime I: closed dynamics for $G=1$	PROVED	C	II.2	Thm 2.3
T33	Non-universality of reach	CORRECTED	C	II.2	Thm 2.11
T34	$G=2$ cannot persist	PROVED	C	II.2	Thm 2.8
L1	Modular escape from any fixed fibre	PROVED	C	II.2	Lem. 2.6
L2	Universal collapse for $G=2,3,6$	PROVED	C	II.2	Thm 2.8
L3	Exact eventual three-step scaling in regime I	PROVED	C	II.2	Thm 2.4
L4	Exact reach counting for $G=1$	PROVED	C	II.2	Thm 2.5
L5	Scope erratum to v005 T33	CORRECTED	C	II.2	Rem. 2.12
L6	Full collapse to $G=1$ (problem)	FALSE	C	II.3	Rem. 3.11
T35	Transition criterion	PROVED	C	II.3	Thm 2.9
T36	Anatomy of a preserving event	PROVED	C	II.3	Thm 2.10
T37	Plateau escape	PROVED	C	II.3	Prop. 2.7
T38	Persistent regime P_g (counterexample to collapse)	PROVED+CHECKED	C	II.3	Thm 3.2
T38a	Existence of P_g for every such g	PROVED+CHECKED	C	II.3	Prop. 3.3
T39	Universal G -collapse is false	PROVED+CHECKED	C	II.3	Thm 3.10
T40	Reach set and growth in P_g	PROVED	C	II.3	Thm 3.8
T41	Status of other g (problem)	CORRECTED	C	II.3	Rem. 3.12
T42	Elementary construction of P_g	PROVED+CHECKED	C	II.3	Thm 3.4
T43	Periodic increment of fibre size	PROVED+CHECKED	C	II.3	Thm 3.7
T44	The gamma-framework	PROVED+CHECKED	C	II.4	Thm 3.1
T45	G_{∞} cannot divide 24	PROVED+CHECKED	C	II.3	Thm 3.13
T46	Persistent regimes with $\gcd(g,6)>1$ (Theorem P^*)	PROVED+CHECKED	C	II.3	Thm 3.5
T47	Persistent orbits live in a bounded gamma-window	PROVED	C	II.4	Thm 3.14
T48	Scaling question (M4a, M4b)	FALSE	C	II.4	Rem. 4.1
T49	The one-step law	PROVED+CHECKED	C	II.4	Thm 4.2
T50	The eight-step theorem ($G=11$)	PROVED+CHECKED	C	II.4	Thm 4.3
T51	Finite-state closure: every persistent orbit scales exactly	PROVED	C	II.4	Thm 4.5
T52	The window theorem	PROVED	C	II.4	Thm 4.6
T53	The absorption criterion	PROVED	C	II.4	Thm 4.7
T54	Carry completeness	PROVED	C	II.4	Thm 4.8
T55	Eventual carries are prime powers	PROVED	C	II.4	Thm 4.9
T56	The decoupled odometer law	PROVED+CHECKED	C	II.4	Thm 4.10
T57	General q -carry theorem	PROVED+CHECKED	C	II.4	Thm 4.12
T58	Corrections to v008	CORRECTED	C	II.4	Rem. 4.14
T59	The carry scheduler	PROVED+CHECKED	C	II.5	Thm 5.1
T60	Single-prime dominance	PROVED+CHECKED	C	II.5	Thm 5.2
T61	The eventual scaling factor is a prime power	PROVED+CHECKED	C	II.5	Thm 5.6
T62	Two saturated clocks have coprime periods	PROVED+CHECKED	C	II.5	Thm 5.8
T63	Corrections to the v010 problem statement	CORRECTED	C	II.5	Rem. 5.9
T67	Exact carry-level predictor	PROVED+CHECKED	C	II.6	Thm 6.2
T64	Minimax selection theorem	PROVED+CHECKED	C	II.6	Thm 6.3
T65	Quotient-residue formula	PROVED+CHECKED	C	II.6	Thm 6.4
T66	Static survivor certificate	PROVED+CHECKED	C	II.6	Thm 6.5
T68	Consequences of static certification	CORRECTED	C	II.6	Thm 6.8
T69	Reset-descent bound (sharp)	PROVED+CHECKED	C	II.7	Thm 7.2
T70	Class-counting entry bound	PROVED+CHECKED	C	II.7	Thm 7.3

ID	Source record title	Source status	Scope	Reg. chap.	Here
T71	Polynomial entry bound	PROVED+CHECKED	C	II.7	Thm 7.4
T72	Dynamic-G effective bound	PROVED+CHECKED	C	II.7	Thm 7.6
T73	Registered corrections and scope (v012)	CORRECTED	C	II.7	Thm 7.7
012.1	Exact transient time / near-sharp bound	OPEN	C	II.7	Prob. F.2
012.2	Static certificate for composite/even g	OPEN	C	II.7	Prob. F.1
012.3	T71 where T67 does not apply	OPEN	C	II.7	Prob. F.3
012.4	Closed-form m_p and survivor for q-carry families	OPEN	C	II.6	Prob. F.4
012.5	Infinitely many carry sizes	OPEN	C	II.4	Prob. F.5
012.6	q=11 class-closure certificate (g=6689)	OPEN	C	II.4	Prob. F.6
010.3	Realisable triples (g, p, Abs_p)	OPEN	C	II.5	Prob. F.7
010.4	Composite/even g sample	OPEN	C	II.5	Prob. F.8
E71	Post-freeze numerical erratum to the T71 data sentence	CORRECTED	C	II.7	Rem. 7.5

Proof dependencies

The table at the end of this appendix gives the registry’s direct dependency list for each record, and Figure B.1 draws the same relations by manuscript chapter, with the main line of the argument highlighted: from the fibre theorem (T1) and the collective rule (CDEF) through monotonicity (T31), the γ -framework (T44), the one-step law (T49), class closure (T51), carry completeness and prime-power carries (T54, T55), the odometer law (T56), single-prime dominance (T60), minimax selection and the exact predictor (T64, T67), the static certificate (T66), and the entry and dynamic- G bounds (T70, T72).

Table of direct dependencies

Direct proof dependencies of the records of this volume, as listed in the registry (records of Volume I appear as dependencies where used).

Record	Depends on (registry dependency list)
CDEF	T8
T17p	T8
T17	CDEF, T17p
T18	T17, T1
T19	T18, T17
T26	T18
T27	T17, T18
T28	T6, CDEF
T30	T22, T17p
T31	T30, CDEF
T32	T30, T31, T22
T33	T31, T32, T30
T34	T30
L1	T30, T31
L2	T30, T31, L1
L3	T32
L4	L3, T31

Record	Depends on (registry dependency list)
L5	T33, L4
L6	L1, L2
T35	T1, T31
T36	T35, T31
T37	L1
T38	T31, T30, T22
T38a	T38
T39	T38, T38a
T40	T38
T41	T34, L2, T38
T42	T38
T43	T38, T46
T44	T30, T31, T35
T45	T44, T30, T35
T46	T38, T44, T31
T47	T44, T37
T48	T44
T49	T44
T50	T49, T44
T51	T49, T47, T44, T32, T31
T52	T49, T44
T53	T49
T54	T53, T44, T47
T55	T54, T53, T51
T56	T54, T55, T53
T57	T49, T54, T53, T56
T58	T50, T57, T54, T48
T59	T53, T54, T55, T56, T51
T60	T59
T61	T60, T56, T54
T62	T56
T63	T60
T67	T53, T54, T49
T64	T56, T60
T65	T56
T66	T64, T65, T67, T52, T55
T68	T66, T51, T60
T69	T54, T44, T37
T70	T51, T52, T54, T55, T60, T66, T67
T71	T55, T64, T67, T56, T65
T72	T31, T44, T37, T69, T70, T32
T73	T68, T66, T69, T70, T71, T72
012.1	T70, T71, T72
012.2	T66, T73
012.3	T71, T67
012.4	T57, T64, T65
012.5	T57
012.6	T51, T57
010.3	T56, T62, T52
010.4	T60
E71	T71

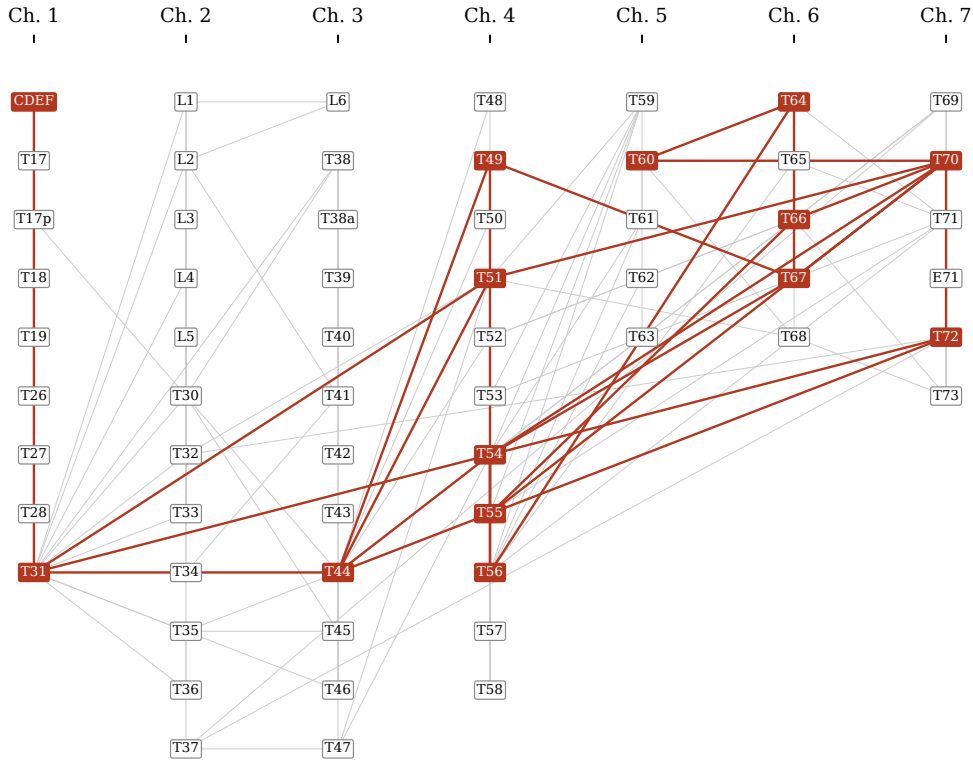


Figure B.1: The proof-dependency graph of the 60 theorem-type records of this volume (nodes arranged in the seven manuscript chapters; 60 nodes, 139 edges within the volume). Red edges: the main line of the argument. Records of earlier chapters appear as dependencies of later ones, never the reverse.

Appendix C Exact arithmetic algorithms and reproducibility

Every numerical statement of this volume is produced by exact integer or rational arithmetic (Python integers and `fractions.Fraction`); no floating-point value enters a statement, a table entry or a certificate. Floating point is used only to draw the figures from exact data. The package contains the full source (`engine.py`, `engine_seeds.py`, `check_volume_ii.py`, `mkfigs2.py`, `gen_tables.py`) and the data files, so that every result can be re-run.

C.1 Algorithms

Direct iteration (`direct`, `Gof`, `Lof`, `fibre`).

Compute $\mathcal{D}(x)$ for $x = a/b$ by testing the divisors d of the square-root part of a with $d \equiv 1 \pmod{b}$ (Lemma 2.1); then G , L , and $C(x) = x + L^2/G$ in `Fraction`. This is the reference implementation: every other algorithm is validated against it.

Residue machine (`mstep`, `normalise`, `state_to_x`).

A persistent state is the triple (g, γ, E) with $x = \gamma L^2/g$, $L = \prod p^{e_p}$. One step applies Theorem 4.2: it computes $w = \gamma + 1$, h , b' , the exponent ceilings s'_p , the set of $d \mid \sigma'$

with $d \equiv 1 \pmod{b'}$ via residues of p^k modulo g , and returns (γ', E') or reports loss of persistence. Cost per step is polynomial in $\sum_p o_p$ over the primes with $p^2 \mid w$ or already in L . Class repetition (Theorem 4.5) is detected by hashing the class vector, which proves eternal periodicity (`min_period`, `cyc_gaps`).

Carry-level predictor (`avail`, `next_M`).

From a reset state, test each candidate $M \geq 2$ (with $M^2 < g$ or $< 4g$) for availability: whether $\prod_{p \mid M} p^{e_p + j_p} \cdot a \equiv 1 \pmod{g}$ for some a in the residue set of the remaining divisors (Theorem 6.2). The smallest available M is the next carry. Cost: $O(\sqrt{g})$ candidates, each a subset-product test of size $\leq \prod (e_q + 1)$, done in the quotient group of Theorem 6.4.

Odometer data (`abs_set`, `pend_j`).

Abs_p , the pending exponent j_p and the worst-case carry m_p of a state (Theorems 4.10, 5.1).

Static certificate (S3)–(S5).

For a reset state E with an apparent survivor r : compute Abs_r , $m = r^{d_r}$, test $m^2 < g$, test that no $M < m$ coprime to r is available (`avail`), and that no prime factor of g is $\leq m^2$. A positive answer proves the whole future of the orbit (Theorem 6.5).

Transient data (`kappa`, `Hclass`, `Kpoly`, `kappa3`).

Exact evaluation of κ , $H(g)$, $K_{\text{poly}}(g)$ and κ_3 from Definition 7.1; the recursion $\mathcal{R}$ of Theorem 7.6 is evaluated by memoised recursion over the divisors of g .

C.2 Verification

The script `check_volume_ii.py` recomputes every statement of the table of checks at the end of this appendix from the definitions, independently of the source-record scripts, on exact finite instances: random seeds, adversarial reset states, the explicit families of Chapters 3 and 4, the sharp witnesses of Table 7.1, and the six composite or even cycles. Its results are consistency tests of finite instances. They do not replace the proofs of the text, and they do not certify a universal statement by sampling; where a universal statement is proved by a finite computation, that is stated in the theorem (class closure, Theorem 4.5; the divisor checks of Theorem 4.12). The run recorded with this volume passes all checks (the count is given at the end of this appendix).

Running the package. `python3 check_volume_ii.py` runs the full check suite with fixed sample sizes and writes `checks_volume_ii.run.json`; `-quick` runs a smaller sample, and `-record` writes the archived `checks_volume_ii.json` instead; `python3 mkfigs2.py` regenerates all figures and `figs/figdata.json` from the registry file and the engine; `python3 gen_tables.py` regenerates the crosswalk, dependency, audit and check tables of the appendices and the two `.tsv` files.

Provenance of the figures. Each figure is computed from an exact orbit or exact table: Figure 1.1 from the orbits listed in its caption; Figure 2.2 from the orbit of 2025/11; Figure 2.1 from all $t \leq 2000$; Figures 4.1–4.4, 4.2 from the residue machine, with the first steps checked against direct iteration; Figure 5.1 from an orbit at $g = 53$; Figure 6.1 from the state of Example 6.7; Figures 7.1, 7.2 from `witnesses.json` and 150 random reset states; Figure 5.2 from 75 exact steps of three orbits; Figure B.1 from the registry dependency lists. The numerical data behind the figures are stored in `figs/figdata.json`.

Table of checks

One line per check; *Result* is **pass** when every instance satisfied the statement.

Tag	Statement checked (exact arithmetic)	Result
T17p	$\Delta_{\text{coll}} = L^2/G$ reduced; $\gamma = Gx/L^2$ integer; $b G$ (4912 seeds)	pass
T31	$D(x) \subset D(C(x))$ and $G(C(x)) \mid G(x)$ (4912 seeds)	pass
T17	plateau law $x_{(n0+k)} = (\gamma_0 + k)L^2/G$ while D is constant (2456 seeds)	pass
T18	growth time $T(x) \leq K_d \leq G d^{2-1}$ for the least d not in D ; $d_{\min} \leq R+2$ (2456 seeds)	pass
T26	minimum of K_d over all d not in D attained at $d \leq \sqrt{(\gamma + k_0)/G} + 1$ (2456 seeds)	pass
T19	$R(x_n)$ non-decreasing and growing along 8 exact orbits (70 steps each)	pass
T27	$n(r+1) \leq n(r) + G_0 (r+2)^{2-1}$ along the same orbits	pass
T28	route-aware orbit $36(k+1)$: $R = 3, 3, 3, 5, 3, 3, 3, 5$ (R drops from 5 to 3)	pass
T30	$b G$, $\gcd(L, G) = 1$, $G = 1 \leq 2$ in $D \leq x$ integer divisible by 4 (4912 seeds)	pass
T32/L3	regime I: $D = \text{divisors of } s$, $G = 1$, entry $\leq 4\kappa_3(t)$, $x_{(n+3)} = 4x_n$, R increment $\tau(\text{odd } s)$ (2249 integers)	pass
L4	reach counting $U(X) = \sum_{q o, q \leq X(1 + \lfloor \log_2(X/q) \rfloor)} - 1$ for the $G = 1$ branch (6 orbits, 75 steps)	pass
T34/L2	$G = 2, 3, 6$ collapse to $G = 1$ within 7, 2, 29 steps (all seeds with these G)	pass
L1/T37	modular escape: a frozen fibre reaches an integer divisible by 4 within $4G$ (L odd) or G (L even, G odd) steps (1037 seeds)	pass
T35	transition criterion d in $D(x') \leq d^2 Y$ and $g/\gcd(g, d-1) \mid Y/d^2$ (1228 seeds, $d < 300$)	pass
T35a	transition identity $G(x') = \gcd(g, d-1)$: d in new)	pass
T36	anatomy of preserving events: coprime to g , fresh prime or carried prime power (281 events)	pass
T38/T42	persistent regime P_g : $x_0 = s^2/g$, $s = (g+1)(g+2)^o$: $G = g$, $x_{(n+3)} = 4x_n$, $L = s_n$ for g in $[5, 7, 11, 13, 17, 19, 23, 25, 29, 35, 49, 55, 65, 77]$	pass
T38h	hypotheses (H1)-(H3) of T38 hold for the T42 seeds	pass
T46	P^* : persistent orbits for $g = 15, 39, 51, 57, 87$, hypothesis (H4) verified	pass
T43	$R(x_{(n+3o)}) - R(x_n) = A_g(m)$ over two complete periods ($g = 13, 19, 25, 55, 15, 39$)	pass
T44	γ framework (a)-(e) on 610 states	pass
T45	orbits with $1 < G \mid 24$ all reach $G = 1$ (544 seeds; G in $[2, 3, 4, 6, 8, 12, 24]$; worst 46 steps)	pass
T47	$\gamma_n \leq 2M$ along persistent orbits of P_g ($M = g$ since L even)	pass
T49	one-step law: residue machine = direct iteration of C (γ, L, G agree on 448 states)	pass
T50	eight-step theorem $g = 11$: $x_{(8k+j)} = (j+1)S_k^2/11$, $x_{(n+8)} = 9x_n$, $R = 7 + 2k$, eight ratios (direct $k \leq 2$, machine $k \leq 399$)	pass
T51	finite-state closure: every sampled orbit becomes class-periodic, $\gamma_{(n+P)} = \gamma_n$, $x_{(n+P)} = \rho^2 x_n$ (60 orbits, 40322 machine states)	pass
T52	window theorem: g odd $\Rightarrow 2 L$, g does not divide $\gamma + 1$, complete carries $M^2 < g$	pass
T54	carry completeness: $\gamma' \leq \gamma_s/4 + 1$; every carry complete after $\gamma = 1$	pass
T55	eventual carries change exactly one prime exponent (prime powers)	pass
T56/T59	scheduler $\min_p p^{(j_p)}$ (odometer law) reproduces every eventual carry	pass
T67	exact carry-level predictor next_M equals the machine at every eventual reset state	pass
T61	single active prime; $P_\gamma = \sum(p^{(2j)} - 1)$, L ratio = p^s (60 cycles)	pass
T57	q -carry theorem instances $q = 3$ ($g = 11, 13$), $q = 5$ ($g = 31, 199$), $q = 7$ ($g = 367$): $x_{(n+q^2-1)} = q^2 x_n$, (Q4) exact	pass

Tag	Statement checked (exact arithmetic)	Result
T57d	q-carry orbits (g=11,13,31) confirmed by direct iteration of C: gamma-cycle, x-scaling, $R=2e_q+1$	pass
T60	at most one odometer fires infinitely often (681472 (system,phase) pairs: 3 odometers, $o \leq 4$, all Abs)	pass
T64	survivor = $\text{argmin}_i p_i^{\text{(maxgap Abs}_i)}$ for every phase (same exhaustive set)	pass
T64r	minimax survivor on 1500 random systems with 2-5 odometers (orders up to 9)	pass
T62	two saturated clocks: Abs_p is tau_p-periodic, tau_p, tau_q coprime (400 cases, prime g)	pass
T65	quotient-residue formula Abs_p=k: $k \bmod t_p$ in S_p, t_p =order of p in $(\mathbb{Z}/g)^*/H$ (600 cases, prime and composite g)	pass
T66	static certificate S3-S5 sound: only r carries and the orbit stays persistent (10 certified states among 500 reset seeds)	pass
T66u	uniqueness: never two primes certified at one state	pass
T68a	prime g: every eventual reset state is statically certified for the active prime (156 states)	pass
T73a	composite/even g=15,39,51,57,68: persistent orbits exist and S5 fails at the reset states (certificate not available); g=39 and 87 follow the same pattern	pass
T69	$\kappa(a)$ =least k with $3a \leq 2 \cdot 4^k + 1$ (thresholds $A_k = (2 \cdot 4^k + 1)/3$; $\kappa(2,10,100,1000,10^6) = 1,2,4,6,11$; $\kappa \leq \text{ceil}(\log_2 a)$)	pass
T69s	bound attained: exact orbits needing exactly $\kappa(\text{gamma0})$ carries for $\kappa=1..11$ (direct iteration of C for $\kappa \leq 8$)	pass
T70v	H(g) and K_poly(g) values for g=11,31,59,101,197,211 (H(211)=24165141422031, K_poly(211)=113903360)	pass
T70/T71	entry index $\leq H(g)-1$ and $\leq K_poly(g)$ on 150 prime-g reset seeds (max observed 191)	pass
T72	dynamic-G chains: (i)-(iv) as inequalities on 1000 orbits / 2235 segments from composite or even G0 (max chain length 4)	pass
T72v	recursive bound R(g0,gamma0) is an explicit finite number (R(12,5), R(30,17))	pass

Appendix D Proof-completeness audit of the volume's records

For each of the 68 records assigned to this volume, the table states where its content stands in the text and how complete the argument given here is. *Full proof*: the statement and a complete proof are in the text, with the hypotheses of the record. *Exact computation*: the claim is a finite exact statement whose certificate is displayed or reproduced by the verifier. *Definition*: the record fixes a rule. *Scope statement* and *Registered correction*: the record states the precise scope of an earlier formulation; it is presented here as a mathematical statement of scope (and, where it corrects a universal claim, the correct theorem is cited). *Open problem*: the record is an open problem and is reported as such. No record is left with only a pointer to an archived proof.

ID	Location	Completeness here	Remark; verification tag
CDEF	Def. 1.1	Definition	the selected collective rule; its consequences are the theorems of the volume
T17p	Prop. 1.2	Full proof	scope A0; check T17p
T17	Prop. 1.4	Full proof	check T17
T18	Thm 1.5	Full proof	check T18

ID	Location	Completeness here	Remark; verification tag
T19	Thm 1.8	Full proof	check T19
T26	Prop. 1.6	Full proof	check T26
T27	Prop. 1.9	Full proof	check T27
T28	Ex. 1.10	Exact computation	exact closed-form orbit, verified by exact iteration; check T28
T30	Thm 2.2	Full proof	scope A0; check T30
T31	Thm 1.3	Full proof	check T31
T32	Thm 2.3	Full proof	check T32/L3
T33	Thm 2.11	Full proof	corrected form; see F.L5
T34	Thm 2.8	Full proof	proved together with F.L2 (Thm 2.8); also reproved by F.T45; check T34/L2
L1	Lem. 2.6	Full proof	check L1/T37
L2	Thm 2.8	Full proof	proved together with F.T34; check T34/L2
L3	Thm 2.4	Full proof	check T32/L3
L4	Thm 2.5	Full proof	check L4
L5	Rem. 2.12	Registered correction	scope of F.T33 stated as Remark 2.12
L6	Rem. 3.11	Scope statement	resolved by F.T39: collapse is false
T35	Thm 2.9	Full proof	check T35, T35a
T36	Thm 2.10	Full proof	check T36
T37	Prop. 2.7	Full proof	check L1/T37
T38	Thm 3.2	Full proof	check T38/T42, T38h
T38a	Prop. 3.3	Full proof	uses Dirichlet's theorem; F.T42 avoids it
T39	Thm 3.10	Full proof	
T40	Thm 3.8	Full proof	
T41	Rem. 3.12	Scope statement	status remark; classification of persistent g is open
T42	Thm 3.4	Full proof	check T38/T42
T43	Thm 3.7	Full proof	check T43
T44	Thm 3.1	Full proof	check T44
T45	Thm 3.13	Full proof	check T45
T46	Thm 3.5	Full proof	check T46
T47	Thm 3.14	Full proof	check T47
T48	Rem. 4.1	Scope statement	universal statement is false; correct statement is F.T51
T49	Thm 4.2	Full proof	check T49
T50	Thm 4.3	Full proof	check T50
T51	Thm 4.5	Full proof	check T51
T52	Thm 4.6	Full proof	check T52
T53	Thm 4.7	Full proof	
T54	Thm 4.8	Full proof	check T54
T55	Thm 4.9	Full proof	check T55
T56	Thm 4.10	Full proof	check T56/T59
T57	Thm 4.12	Full proof	check T57, T57d
T58	Rem. 4.14	Registered correction	scope statements in Rem. 4.14
T59	Thm 5.1	Full proof	check T56/T59
T60	Thm 5.2	Full proof	check T60
T61	Thm 5.6	Full proof	check T61
T62	Thm 5.8	Full proof	check T62
T63	Rem. 5.9	Registered correction	scope statement in Rem. 5.9
T67	Thm 6.2	Full proof	check T67
T64	Thm 6.3	Full proof	check T64, T64r
T65	Thm 6.4	Full proof	check T65
T66	Thm 6.5	Full proof	prime g , or no prime factor of $g \leq m^2$; check T66, T66u
T68	Thm 6.8	Full proof	part (a) proved for prime g ; see F.T73; check T68a

ID	Location	Completeness here	Remark; verification tag
T69	Thm 7.2	Full proof	sharp; witnesses for $\kappa \leq 11$ (Table 7.1); check T69, T69s
T70	Thm 7.3	Full proof	clause (c) for prime g ; check T70/T71, T70v
T71	Thm 7.4	Full proof	prime g , or whenever F.T67 applies; check T70/T71
T72	Thm 7.6	Full proof	counts carry events; steps via plateau bound; check T72, T72v
T73	Thm 7.7	Registered correction	scope of certification; effectiveness; check T73a
012.1	Prob. F.2	Open problem	near-sharp transients
012.2	Prob. F.1	Open problem	composite/even certification
012.3	Prob. F.3	Open problem	composite g
012.4	Prob. F.4	Open problem	closed form of m_p
012.5	Prob. F.5	Open problem	all q
012.6	Prob. F.6	Open problem	class closure at $q = 11$
010.3	Prob. F.7	Open problem	realisable triples
010.4	Prob. F.8	Open problem	composite/even sample
E71	Rem. 7.5	Registered correction	numerical clause only

Appendix E Mathematical contributions and relation to existing theory

This appendix places the results of the volume in their mathematical setting. The techniques are classical and the objects are new: the contribution of the UNNS theory is the identification of the collective map C , of the coordinates (γ , exponent state, absorption sets) in which its dynamics become finite, and the theorems that follow. The table pairs each group of results with the established background it uses.

Results	Established tools used	What the theory establishes
Fibre growth (Thms 1.5, 1.8, Prop. 1.9)	valuations, gcd/lcm, divisor counting	unbounded multiplicity of the collective orbit, an explicit growth-time bound, and a cubic-root lower rate
Regime I (Thms 2.3, 2.4, 2.5)	squarefree decomposition; iteration of $t \mapsto \text{sqf}(t+1)$	closed dynamics at $G = 1$, exact three-step scaling and exact reach counting
Exclusions (Thms 2.8, 3.13, Lem. 2.6)	residues; descent on positive integers	G_∞ can never divide 24, by a descent argument on γ
Persistent regimes (Thms 3.2, 3.4, 3.5, Prop. 3.3)	orders of 2 modulo g ; Dirichlet's theorem (only for Prop. 3.3)	explicit infinite families with $G_\infty = g > 1$, including $3 \mid g$: universal collapse is false
γ -framework, window (Thms 3.1, 3.14)	integrality; congruences modulo 8, 3, 24	an integer coordinate in which persistent orbits live in a bounded window
Class closure, exact scaling (Thm 4.5)	finite deterministic systems are eventually periodic; structure of $(\mathbb{Z}/g)^\times$	every orbit eventually satisfies $x_{n+P} = \rho^2 x_n$; the finite-state data are identified
Carry (Thms 4.7–4.10)	cyclic subgroups, orders of primes modulo g	carry completeness, prime-power carries, the odometer law
q -carry family (Thm 4.12)	cyclic quotient $\langle q \rangle, \langle 2 \rangle H$	an explicit orbit family with cycle $1, \dots, q^2 - 1$; realised for $q = 3, 5, 7$

Results	Established tools used	What the theory establishes
Single-prime dominance (Thms 5.1, 5.2, 5.6)	extremal (worst-case) argument on cyclic odometers	exactly one prime carries; ρ is a prime power; logarithmic reach (Cor. 5.7)
Selection and certificates (Thms 6.2–6.8)	quotient groups $(\mathbb{Z}/g)^\times/H$; minimax	survivor decided by a finite residue test at one reset state
Effective bounds (Thms 7.2–7.6)	iteration of $a \mapsto \lfloor a/4 \rfloor + 1$; pigeonhole on classes; epoch decomposition	computable bounds on the transient from (g_0, γ_0)

Established background. The following facts are used as standard: the unit group $(\mathbb{Z}/p)^\times$ of a prime is cyclic, and the order and subgroup structure of a cyclic group are those of $\mathbb{Z}/n$ [15, 16]; Dirichlet’s theorem on primes in arithmetic progressions [17], used once (Proposition 3.3) and avoided in Theorem 3.4; the pigeonhole principle for a deterministic map on a finite set; valuations and the characterisation of gcd and lcm through them. None of these is specific to the UNNS Substrate.

Results about UNNS structures. What is specific to the Substrate is the following chain, each link a theorem of this volume. The collective rule C is chosen (Definition 1.1); once chosen, its orbits have unbounded multiplicity (Theorem 1.8); the coordinate γ and the one-step law reduce the dynamics to residues (Theorems 3.1, 4.2); every orbit is eventually exactly self-similar (Theorem 4.5); the self-similar regime is a single odometer (Theorem 5.2); which odometer is decided by finite arithmetic (Theorems 6.3, 6.5); and the time to reach it is bounded effectively (Theorem 7.6). The distinction between existence (eventual exact scaling), structure (one prime, one carry word) and effectiveness (computable transient bounds) is the spine of the volume.

Open directions. The scope of certification beyond prime g , sharper transient bounds, the classification of the g that admit persistent orbits, and the dynamics of collective rules other than C are discussed in Appendix F. The collective rule is a selected rule: Volume I shows that the interaction axioms considered there do not select a unique rule, so every statement of this volume is a statement about C .

Appendix F Open problems and further mathematical development

The records below are the open problems of the registry for this volume. Each is stated with its precise setting, so that progress can be measured against the theorems of the text.

Open problem F.1 (Static certification for composite or even g). SOURCE RECORD: O12.2. Replace hypothesis (S5) of Theorem 6.5 by a condition on the plateau steps with $\gcd(g, w) > 1$, so that a finite residue test certifies the survivor and the whole future for composite and even g . Known: the survivor exists (Theorem 5.2), the class cycle is entered within $H(g)$ carries (Theorem 7.3(a)), and all reset states of the six cycles $g = 15, 39, 51, 57, 68, 87$ fail (S5) (Theorem 7.7).

Open problem F.2 (Exact or near-sharp transient time). SOURCE RECORD: O12.1. Find the exact transient time, or a bound within a small factor of the observed entry index. Observed

entry indices are of order 10^2 (the recorded run has a maximum of 208 at $g = 211$; Figure 7.2 gives the sample computed for this volume), while the proved bounds $\min(H, K_{\text{poly}})$ exceed 10^8 there. The route suggested by the proof of Theorem 7.4 is to bound the epochs by the structure of the saturated odometers rather than by their number of states.

Open problem F.3 (The polynomial bound beyond the carry-level predictor). SOURCE RECORD: O12.3. Prove a polynomial entry bound for composite g outside the scope of Theorem 6.2 (where $\gcd(g, w) > 1$ can occur on the way).

Open problem F.4 (Closed form of the worst-case carry and the survivor for q -carry families). SOURCE RECORD: O12.4. Give m_p and the survivor of the families of Theorem 4.12 in closed form from (g, q) alone.

Open problem F.5 (Infinitely many carry sizes). SOURCE RECORD: O12.5. Decide whether (Q1)–(Q4) of Theorem 4.12 are solvable for every odd prime q . Eleven exact instances with $q \leq 37$ are recorded. The conditions are finitely many non-membership conditions in a cyclic quotient of $(\mathbb{Z}/g)^\times$, and a Chebotarev or Kummer-type argument appears plausible; it has not been carried out.

Open problem F.6 (The case $q = 11$, $g = 6689$). SOURCE RECORD: O12.6. Complete the class-closure certificate for $q = 11$, $g = 6689$; about 50 000 machine steps are needed.

Open problem F.7 (Realisable triples). SOURCE RECORD: O10.3. Determine which triples (g, p, Abs_p) are realised by actual orbits. Theorem 5.8 and the window $c^2 < g$ restrict them; a complete characterisation is not known.

Open problem F.8 (Composite and even samples). SOURCE RECORD: O10.4. Theorems 5.1 and 5.2 do not use primality of g , but the numerical sample is almost entirely prime g . A systematic sample of composite and even g would test the picture of Chapter 5 beyond the six known cycles.

Further directions

- *Classification of persistent values.* Which g admit persistent orbits? The answer is known for $g \mid 24$ (none), for $\gcd(g, 6) = 1$ (all $g \geq 5$), and in the instances $g = 15, 39, 51, 57, 87, 68$ (Remark 3.12).
- *Scope of the polynomial bound.* The proof of Theorem 7.4 counts epochs of the saturated odometers; a refinement that exploits the periodicity of Abs_p (Theorem 5.8) may reduce K_{poly} substantially.
- *Other collective grammars.* The theory of this volume concerns the selected rule C . The same programme (a coordinate in which the dynamics becomes finite, a scheduler, a certificate and effective bounds) can be asked for other collective rules; Volume I shows that the interaction axioms considered there do not single out one.

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