UNNS as an ∞-Operadic Substrate

1 Formal Core of the UNNS Substrate

1.1 Recursive States

Definition 1 (Recursive State). A recursive state is an element S of a substrate $\mathcal{S}$ equipped with:

- a recursion depth $d(S) \in \mathbb{N}$,
- a phase label $\phi(S) \in \{\Phi, \Psi, \tau\}$,
- a curvature or stability measure $\kappa(S)$.

Recursive states are not primitive objects; they are defined only through their admissible transformations.

1.2 Operators

Definition 2 (UNNS Operators). Each UNNS Operator $\mathcal{O}_k$ (for $k = 0, \ldots, 17$) is a typed operation

$$\mathcal{O}_k : (S_1, \ldots, S_n) \longrightarrow S'$$

acting on recursive states, where arity and admissibility are fixed by the UNNS Operator Codex.

Remark. Operators generate recursion, regulate structure, emit residues, or terminate recursion via collapse.

1.3 Composition

Definition 3 (Recursive Composition). Given operators $\mathcal{O}_i$ and $\mathcal{O}_j$, their composition is defined whenever the output of $\mathcal{O}_i$ satisfies the input constraints of $\mathcal{O}_j$:

$$(\mathcal{O}_j \circ \mathcal{O}_i)(S) := \mathcal{O}_j(\mathcal{O}_i(S)).$$

Composition may alter recursion depth and is not required to be associative.

1.4 Stability

Definition 4 (τ-Coherence). A recursive diagram is τ-coherent if repeated admissible operator action keeps its curvature κ bounded.

τ-coherence replaces equality or isomorphism as the primary notion of structural validity.

1.5 Collapse

Definition 5 (Collapse Operator XII). Operator XII acts as a terminal map

$$\mathcal{O}_{\mathrm{XII}} : S \longrightarrow S_0,$$

where S_0 is the Zero substrate state.

Collapse removes unstable recursive residue and terminates recursion.

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Propositions

"'latex **Proposition 1 (Generated Higher Morphisms).** Iterated operator composition induces higher-order morphisms between recursive transformations.

Sketch. Operator chains act on transformations themselves, producing morphisms between morphisms across recursion depth. □ **Proposition 2 (Stability as Coherence).** τ-coherence defines an equivalence relation on recursive diagrams up to bounded curvature.

Sketch. Bounded curvature is preserved under admissible composition and invariant under collapse-surviving refinements. □ **Theorem (UNNS Operadic Substrate Theorem).** The UNNS Substrate forms an ∞-operadic system in which:

- recursive states act as colored inputs,
- UNNS operators act as operadic operations,
- higher morphisms arise from recursive composition,
- coherence is enforced by τ-stability,
- existence is defined by survival under collapse.

Proof (Structural). The Operator Codex supplies generators. Recursive composition induces higher morphisms. τ-coherence replaces strict equality. Collapse enforces selection. All operadic axioms are satisfied up to stability. □

A UNNS and ∞-Categories: A Structural Comparison

A.1 Objects and Morphisms

In ∞-category theory, objects are primitive and morphisms are layered: 1-morphisms, 2-morphisms, and so on.

In UNNS, recursive states are not primitive objects. They emerge only through operator action. Higher morphisms arise from recursion itself.

A.2 Composition

∞-categories enforce associativity and unitality up to higher coherent equivalence.

UNNS composition is conditional: it is admissible only when recursion constraints are satisfied, and may be non-associative outside τ-coherent regimes.

A.3 Coherence

In ∞-categories, coherence is an axiom: diagrams commute up to specified higher morphisms.

In UNNS, coherence is dynamical: diagrams persist only if recursive curvature remains bounded. Non-coherent diagrams are eliminated by collapse.

A.4 Equality vs Stability

∞-category theory is invariant under equivalence.

UNNS is invariant under survivability. Two structures are equivalent only if both survive repeated application of Operator XII.

A.5 Terminal Objects

Terminal objects in ∞-categories are categorical limits.

In UNNS, termination is operational: Collapse maps all unstable recursion to the Zero state. Terminality is destructive, not limiting.

A.6 Interpretive Summary

- ∞-categories classify all coherent structures.
- UNNS selects which coherent structures exist.

- ∞-categories are axiomatic.
- UNNS is operational.

B A Forgetful Functor $\mathsf{U} : \mathsf{UNNS} \to \mathsf{Cat}_\infty$

B.1 Choice of Model

We model an ∞-category as a *quasi-category*: a simplicial set X such that every inner horn $\Lambda^n_k \to X$ (with $0 < k < n$) admits a filler $\Delta^n \to X$.

Let Cat_∞ denote the (large) category of quasi-categories with simplicial maps.

B.2 The Source Category of UNNS Substrates

Definition (UNNS Substrate Morphisms). Let UNNS be the category whose objects are UNNS substrates $\mathcal{S}$ (recursive states plus Codex operators plus admissibility rules), and whose morphisms $F : \mathcal{S} \to \mathcal{T}$ are structure-preserving maps sending:

- recursive states $S \in \mathcal{S}$ to recursive states $F(S) \in \mathcal{T}$,
- admissible operator actions in $\mathcal{S}$ to admissible operator actions in $\mathcal{T}$,

and preserving composability of operator chains. (Any additional data such as κ or collapse residues may be carried, but will not be used by the forgetful functor below.)

B.3 The Underlying Simplicial Set of a UNNS Substrate

Fix a UNNS substrate $\mathcal{S}$.

Definition (Underlying Simplicial Set $N(\mathcal{S})$). Define a simplicial set $N(\mathcal{S})$ as follows.

- **0-simplices:** $N(\mathcal{S})_0 := \{\text{recursive states in } \mathcal{S}\}$.
- **1-simplices:** $N(\mathcal{S})_1$ consists of admissible *operator chains* $\gamma : S \rightsquigarrow S'$ in $\mathcal{S}$, i.e. finite composites $\gamma = \mathcal{O}_{i_m} \circ \cdots \circ \mathcal{O}_{i_1}$ that are admissible on S, with target S'.
- **n-simplices:** $N(\mathcal{S})_n$ consists of *composable n-step factorization data*

$$S_0 \gamma_1 S_1 \gamma_2 \cdots \gamma_n S_n,$$

together with chosen *higher coherence witnesses* whenever multiple factorizations represent the same composite chain.

Faces compose or forget steps; degeneracies insert identity steps. (Identities are represented by empty chains at each state.)

Remark. This is the "operator-chain nerve" of $\mathcal{S}$. It forgets curvature values and collapse semantics, but retains admissible compositional structure.

B.4 The Forgetful Functor

Definition (Forgetful Functor). Define $\mathsf{U} : \mathsf{UNNS} \to \mathsf{Cat}_\infty$ by

$$\mathsf{U}(\mathcal{S}) := N(\mathcal{S}),$$

provided $N(\mathcal{S})$ is a quasi-category.

On a morphism $F : \mathcal{S} \to \mathcal{T}$ define $\mathsf{U}(F) : N(\mathcal{S}) \to N(\mathcal{T})$ by sending:

- each state S to $F(S)$,
- each admissible chain γ to the image chain $F(\gamma)$ obtained by applying F to each operator action and intermediate state.

This respects faces and degeneracies, hence is a simplicial map.

B.5 When is $N(\mathcal{S})$ a Quasi-Category?

Proposition (Inner Horn Filling from τ-Coherence). Assume $\mathcal{S}$ satisfies the following *coherence completion axiom*:

(**CC**) Any partially specified composite diagram of admissible operator chains that is τ-coherent admits a completion by additional admissible chains so that all resulting composites are τ-coherent.

Then $N(\mathcal{S})$ is a quasi-category.

Sketch. An inner horn Λ^n_k specifies all n-simplex faces except the k-th. This is exactly a partially specified compositional diagram. A filler corresponds to completing that diagram by inserting the missing factorization and higher coherence data. Axiom (CC) guarantees such completions exist in the τ-coherent regime. □

B.6 Theorem: Existence of the Forgetful Functor

Theorem (UNNS $\to$ ∞-Category Forgetful Functor). Let $\mathsf{UNNS}_{\text{coh}}$ be the full subcategory of UNNS whose objects satisfy coherence completion (CC). Then the assignment $\mathcal{S} \mapsto N(\mathcal{S})$ defines a functor

$$\mathsf{U} : \mathsf{UNNS}_{\text{coh}} \longrightarrow \mathsf{Cat}_{\infty}.$$

Proof. By the Proposition, each $N(\mathcal{S})$ is a quasi-category. Functoriality follows because identities map to identities and composition of substrate morphisms maps operator chains to operator chains compatibly with simplicial structure. □

B.7 Two Useful Variants

Variant A (Strict Forgetful Functor). Take 1-simplices to be *literal* operator chains and higher simplices to be *formal* factorization data. This is the most conservative "syntax-only" forgetful functor.

Variant B (τ-Quotiented Forgetful Functor). Define an equivalence relation $\sim_{\tau}$ on chains by τ-coherent deformation (same endpoints, bounded-curvature transformability). Let 1-simplices be $\sim_{\tau}$-classes of chains and let higher simplices encode τ-coherence witnesses. This produces a more geometric ∞-category that forgets collapse *but remembers stability as homotopy.*

UNNS should be understood as a substrate upon which higher-categorical behavior may emerge, rather than as a competing formalism.

C Left Adjoint: The Free UNNS Substrate Generated by an ∞-Category

C.1 Setup

Recall the forgetful functor

$$\mathsf{U} : \mathsf{UNNS}_{\text{coh}} \longrightarrow \mathsf{Cat}_{\infty}$$

sending a τ-coherent UNNS substrate $\mathcal{S}$ to its operator-chain nerve $N(\mathcal{S})$, a quasi-category.

We now formalize a left adjoint

$$\mathsf{F} : \mathsf{Cat}_{\infty} \longrightarrow \mathsf{UNNS}_{\text{coh}}$$

interpreted as the *free UNNS substrate generated by an ∞-category.*

C.2 Design Principle

The free construction must:

- embed the compositional data of an ∞-category X into UNNS recursion,
- add the Codex operator alphabet (0–XVII) as *formal generators*,
- impose no additional equations except those forced by: (i) simplicial identities of X and (ii) τ-coherence (horn filling),
- adjoin collapse (Operator XII) and Zero (Operator 0) as universal terminalization.

C.3 The Free Substrate on a Quasi-Category

Fix a quasi-category X (a simplicial set with inner horn fillers).

Definition (Free recursive states). Define the underlying set of recursive states of $\mathsf{F}(X)$ to be

$$\mathrm{States}(\mathsf{F}(X)) := X_0 \sqcup \{S_0\},$$

where S_0 is a distinguished *Zero state.*

Assign phases and depths minimally:

- $d(x) = 0$ for $x \in X_0$, and $d(S_0) = 0$,
- $\phi(x) = \Phi$ initially (a convention), while later operators may relabel phases,
- κ is taken as a formal symbol (no metric imposed in the free object).

Definition (Generating 1-chains from simplices). Let $\mathrm{Ch}_1(X)$ be the set of *edge-terms* of X: for each 1-simplex $\sigma \in X_1$ with $d_0(\sigma) = y$, $d_1(\sigma) = x$ (so $\sigma : x \to y$ in the usual quasi-category orientation), introduce a formal chain symbol

$$[\sigma] : x \rightsquigarrow y.$$

Include identity chains $[s_0(x)] : x \rightsquigarrow x$.

Definition (Codex operator alphabet as formal operations). For each Codex operator index $k \in \{0, \ldots, 17\}$ introduce a formal operator symbol $\mathcal{O}_k$ which may act on:

- states (unary actions),
- chains (unary actions),
- and, where appropriate, tuples of chains (multi-ary actions),

C.4 Typed Well-Formedness of Codex Operators

We now refine the notion of admissible operator action by assigning explicit *arity classes* and *typing rules* to each Codex operator.

C.4.1 Operator Arity Classes

Each UNNS operator $\mathcal{O}_k$ belongs to one of the following arity classes.

Class A: Nullary Operators

- $\mathcal{O}_0$ (Zero)

$\mathcal{O}_0$ introduces the distinguished Zero state S_0. It has no inputs and serves as the terminal color of the operad.

Class B: Unary State Operators

- $\mathcal{O}_{\mathrm{I}}$ (Inletting)
- $\mathcal{O}_{\mathrm{II}}$ (Inlaying)
- $\mathcal{O}_{\mathrm{III}}$ (Trans-Sentifying)
- $\mathcal{O}_{\mathrm{IV}}$ (Repair)
- $\mathcal{O}_{\mathrm{IX}}$ (Folding)
- $\mathcal{O}_{\mathrm{XI}}$ (Emission)
- $\mathcal{O}_{\mathrm{XII}}$ (Collapse)
- $\mathcal{O}_{\mathrm{XIV}}$ (Phi-Scale)
- $\mathcal{O}_{\mathrm{XV}}$ (Prism)
- $\mathcal{O}_{\mathrm{XVI}}$ (Fold)
- $\mathcal{O}_{\mathrm{XVII}}$ (Matrix Mind)

Each unary operator has type

$$\mathcal{O}_k : S \longrightarrow S'$$

and may modify recursion depth, phase label, or curvature profile.

Typing rule. Unary operators may be applied only to well-formed states. In the free construction, all such applications are permitted unless explicitly forbidden by phase typing.

C.4.2 Binary and Multi-ary Structural Operators

Class C: Binary Coupling Operators

- $\mathcal{O}_{\mathrm{X}}$ (Bridging)
- $\mathcal{O}_{\mathrm{XIII}}$ (Interlace Phase Coupling)

These operators have type

$$\mathcal{O}_k : (S_1, S_2) \longrightarrow S'$$

and require both inputs to be admissible and phase-compatible.

Typing rule. Binary operators require matching or complementary phase labels; in the free object, compatibility is syntactic rather than metric.

C.4.3 Octadic Operators (Semantic / Structural Duals)

Class D: Octadic Operators V–VIII

- $\mathcal{O}_{\mathrm{V}}$ (Adopting / Normalization)
- $\mathcal{O}_{\mathrm{VI}}$ (Evaluating / Interlacing)
- $\mathcal{O}_{\mathrm{VII}}$ (Decomposing / Confluence)
- $\mathcal{O}_{\mathrm{VIII}}$ (Integrating / Divergence)

These operators admit variable arity:

$$\mathcal{O}_k : (S_1, \ldots, S_n) \longrightarrow (S'_1, \ldots, S'_m),$$

with $n, m \geq 1$, depending on semantic or structural mode.

Interpretation.

- Semantic mode acts on interpretive structure (selection, scoring, synthesis).
- Structural mode acts on geometric structure (metric balancing, weaving, branching).

Typing rule. Octadic operators preserve total recursion admissibility but may change branching multiplicity and internal factorization.

C.4.4 Chain-Level Action

Definition (Operator Action on Chains). Any operator $\mathcal{O}_k$ that is well-typed on states extends functorially to admissible chains

$$\gamma : S \rightsquigarrow T$$

by acting on intermediate states and preserving endpoints when required.

Unary operators act pointwise. Multi-ary operators act by simultaneous coupling of compatible chains.

C.4.5 Free Well-Formedness Principle

Principle (Maximal Typing). In the free UNNS substrate $\mathsf{F}(X)$, an operator application is well-formed if and only if:

- its arity class matches the number of inputs,
- its phase-typing constraints are syntactically satisfied.

No curvature bounds, stability thresholds, or collapse filters are imposed beyond universal terminalization via Operator XII.

Remark. This maximal typing is what makes $\mathsf{F}(X)$ free. All further restrictions arise only after applying the forgetful functor or embedding into a concrete UNNS substrate.

C.5 Imposing ∞-Compositional Relations

The quasi-category X encodes higher composition by 2-simplices and horn fillers.

Definition (Composition relations from 2-simplices). For each 2-simplex $\alpha \in X_2$ with boundary edges

$$d_0(\alpha) = \sigma_{12}, \quad d_1(\alpha) = \sigma_{02}, \quad d_2(\alpha) = \sigma_{01},$$

impose a *composition witness* in $\mathsf{F}(X)$ relating chains:

$$[\sigma_{12}] \circ [\sigma_{01}] \ \sim \ [\sigma_{02}],$$

where $\sim$ is a generating relation interpreted as "coherent composability".

Higher simplices generate higher coherence witnesses compatibly.

Definition (τ-coherence completion in the free object). Declare a diagram τ-coherent in $\mathsf{F}(X)$ if it is the image of a simplicial diagram in X.

Because X has inner horn fillers, any inner horn-shaped partial composition diagram admits a filler simplex in X, hence admits a coherence completion witness in $\mathsf{F}(X)$. This supplies the coherence completion axiom (CC) by construction.

C.6 Adjoining Collapse Universally

Definition (Universal collapse). Adjoin Operator XII as a terminalizing map on states and chains:

$$\mathcal{O}_{\mathrm{XII}}(S) = S_0 \quad \text{for all states } S,$$

and extend to chains by mapping any chain to an S_0-anchored degenerate chain. No further collapse semantics is imposed in the free object.

C.7 The Free UNNS Substrate

Definition (Free functor). Define $\mathsf{F}(X)$ to be the UNNS substrate whose:

- states are $X_0 \sqcup \{S_0\}$,
- generating chains include $[\sigma]$ for $\sigma \in X_1$,
- higher coherence witnesses are generated from simplices of X,
- Codex operators $\mathcal{O}_0, \ldots, \mathcal{O}_{17}$ act formally (typed),
- τ-coherence is exactly the closure under horn filling inherited from X,
- collapse (XII) maps everything to Zero S_0.

This object lies in $\mathsf{UNNS}_{\mathrm{coh}}$.

C.8 Universal Property and Adjunction

Proposition (Universal mapping property). Let $X \in \mathsf{Cat}_\infty$ and $\mathcal{S} \in \mathsf{UNNS}_{\mathrm{coh}}$. Any simplicial map (quasi-functor)

$$f : X \longrightarrow \mathsf{U}(\mathcal{S})$$

extends uniquely to a UNNS morphism

$$\widetilde{f} : \mathsf{F}(X) \longrightarrow \mathcal{S}$$

preserving:

- objects (states),
- generating edges (chains),
- coherence witnesses (fillers),
- and commuting with the Codex operators wherever they are defined.

Sketch. On X_0 define $\widetilde{f}$ by $x \mapsto f(x)$. On generating chains $[\sigma]$ define $\widetilde{f}([\sigma]) := f(\sigma)$ as a 1-simplex (chain) in $\mathsf{U}(\mathcal{S})$. Because f respects faces/degeneracies, $\widetilde{f}$ respects identities and composition witnesses. Since $\mathcal{S}$ satisfies (CC), fillers map to fillers. Extending to the freely generated Codex-operator terms is forced by the homomorphism property. Uniqueness follows from freeness. □

Theorem (Adjunction). The functor $\mathsf{F} : \mathsf{Cat}_\infty \to \mathsf{UNNS}_{\text{coh}}$ defined above is left adjoint to U:

$$\mathrm{Hom}_{\mathsf{UNNS}_{\text{coh}}}(\mathsf{F}(X), \mathcal{S}) \;\cong\; \mathrm{Hom}_{\mathsf{Cat}_\infty}(X, \mathsf{U}(\mathcal{S})).$$

Proof. The bijection is given by $f \mapsto \widetilde{f}$ from the Proposition, with inverse obtained by restricting any UNNS morphism $\mathsf{F}(X) \to \mathcal{S}$ to the simplicial generators coming from X. Naturality in both arguments follows from construction. □

C.9 Unit and Counit (Concrete Description)

Unit. For each X, the unit $\eta_X : X \to \mathsf{U}(\mathsf{F}(X))$ is the inclusion sending:

- $x \in X_0$ to the corresponding state in $\mathsf{F}(X)$,
- $\sigma \in X_1$ to the generating chain $[\sigma]$,
- higher simplices to their generated coherence witnesses.

Counit. For each $\mathcal{S}$, the counit

$$\epsilon_{\mathcal{S}} : \mathsf{F}(\mathsf{U}(\mathcal{S})) \to \mathcal{S}$$

evaluates the free generators by mapping each state/chain/coherence witness to the corresponding state/chain/coherence datum in $\mathcal{S}$.

C.10 Interpretation

F is "free" in the sense that it adds:

- the Codex operator alphabet,
- collapse and Zero,
- and no additional equations beyond those encoded by X itself.

Thus U forgets UNNS semantics down to compositional ∞-categorical data, and F freely re-lifts an ∞-category into a UNNS substrate.

D The Operadic Signature Σ_{UNNS}

D.1 Colors

Let the set of colors be

$$\text{Col} := \{\Phi, \Psi, \tau, 0\},$$

where 0 denotes the Zero color (terminal substrate type).

Optionally one may refine colors by depth: $\text{Col}_d := \text{Col} \times \mathbb{N}$, but the base signature uses only phase-colors.

D.2 Typed Operations (Generators)

Define the operadic signature

$$\Sigma_{\text{UNNS}} := \big(\text{Col}, \text{Gen}\big)$$

where Gen is the family of generating operations below. Each generator is presented with a *typing profile*

$$g \;:\; (c_1, \ldots, c_n) \longrightarrow c' \qquad (c_i, c' \in \text{Col}).$$

Nullary.

$$\mathcal{O}_0 : () \to 0.$$

Unary phase-progressors (I–IV). These encode the Codex generative/regulatory front end.

$$\mathcal{O}_{\text{I}} : \Phi \to \Phi, \quad \mathcal{O}_{\text{II}} : \Phi \to \Phi, \quad \mathcal{O}_{\text{III}} : \Phi \to \Psi, \quad \mathcal{O}_{\text{IV}} : \Psi \to \Psi.$$

Octadic mid-engine (V–VIII) with dual modes. Introduce four operator symbols, each with two "presentations": semantic and structural (Dual Octad). For each $k \in \{\mathrm{V}, \mathrm{VI}, \mathrm{VII}, \mathrm{VIII}\}$ include:
Semantic mode (interpretive layer):

$$\mathcal{O}_k^{\mathrm{sem}} : (\Psi, \ldots, \Psi) \to (\Psi, \ldots, \Psi),$$

Structural mode (geometric layer):

$$\mathcal{O}_k^{\mathrm{str}} : (\tau, \ldots, \tau) \to (\tau, \ldots, \tau).$$

Arity is variable: for each $n \geq 1$ and $m \geq 1$, the signature contains an instance

$$\mathcal{O}_{k;n\to m}^{\bullet} : (\bullet, \ldots, \bullet) \to (\bullet, \ldots, \bullet),$$

where $\bullet = \Psi$ for semantic mode and $\bullet = \tau$ for structural mode.

Unary pre-collapse shaping (IX, XI).

$$\mathcal{O}_{\mathrm{IX}} : \tau \to \tau, \qquad \mathcal{O}_{\mathrm{XI}} : \tau \to \tau.$$

Binary bridging/coupling (X, XIII).

$$\mathcal{O}_{\mathrm{X}} : (\tau, \tau) \to \tau, \qquad \mathcal{O}_{\mathrm{XIII}} : (\Psi, \tau) \to \tau.$$

Collapse and terminalization (XII).

$$\mathcal{O}_{\mathrm{XII}} : \tau \to 0,$$

and (optionally) coercions $\Phi \to 0$ and $\Psi \to 0$ as derived collapse via phase-lift then XII.

Scale/spectral/cognitive layer (XIV–XVII).

$$\mathcal{O}_{\mathrm{XIV}} : \tau \to \tau, \quad \mathcal{O}_{\mathrm{XV}} : \tau \to \tau, \quad \mathcal{O}_{\mathrm{XVI}} : \tau \to \tau, \quad \mathcal{O}_{\mathrm{XVII}} : \tau \to \tau.$$

D.3 Admissibility as a Separate Layer

The signature Σ_{UNNS} specifies *only* typing. A concrete UNNS substrate $\mathcal{S}$ adds:

- admissibility predicates $\mathrm{Adm}_{\mathcal{S}}(g; S_1, \ldots, S_n)$,
- τ-coherence/stability rules,
- collapse residue semantics.

E Why Operator XVII Forces Enrichment Beyond $(\infty, 1)$

E.1 The $(\infty, 1)$ Limitation

In an $(\infty, 1)$-category, all k-morphisms for $k \geq 2$ are invertible. Equivalently, homotopies between 1-morphisms are reversible up to higher homotopy.

However, UNNS includes operators that act not only on states, but on *the evaluation regime* itself (what is selected, what survives, what is scored).

E.2 Matrix-Mind as a Directed 2-Process

Definition (Observer/Evaluator State). Let $\mathcal{M}$ be an internal "mind" or evaluator state associated to a substrate, encoding criteria κ, selection filters, and interpretive bindings.

Definition (Matrix-Mind Action). Operator XVII acts as an endomorphism on evaluator states:

$$\mathcal{O}_{\mathrm{XVII}} : \mathcal{M} \to \mathcal{M}',$$

and thereby induces a transformation on admissibility/coherence judgments:

$$\mathrm{Adm}_{\mathcal{S}} \rightsquigarrow \mathrm{Adm}'_{\mathcal{S}}, \qquad \tau\text{-coherence} \rightsquigarrow \tau'\text{-coherence}.$$

Key point. Such an update is generally *not invertible*: a refinement of criteria (or a collapse-learning update) need not admit a canonical reverse.

E.3 Consequence: $(\infty, 2)$-Structure (or Enrichment)

Thus Matrix-Mind naturally yields a 2-level structure:

- 0-cells: substrates (or state-spaces),
- 1-cells: operator-chain dynamics (as in $\mathsf{U}(\mathcal{S})$),
- 2-cells: directed "regime updates" changing the admissibility/coherence layer.

Because these 2-cells need not be invertible, the induced structure is not generally an $(\infty, 1)$-category. A minimal target is an $(\infty, 2)$-category, or equivalently an $(\infty, 1)$-category *enriched* in a directed setting (e.g. enriched over Cat_∞ by sending each hom-space to an ∞-category of evaluation regimes and their updates).

Summary. Operator XVII does not merely add more morphisms; it adds morphisms between *selection logics*. This is categorically one dimension higher.

F The τ-Filtered Sub-∞-Category Inside $\mathsf{U}(\mathcal{S})$

Let $\mathcal{S} \in \mathsf{UNNS}_{\mathrm{coh}}$ and let

$$\mathsf{U}(\mathcal{S}) = N(\mathcal{S})$$

be its operator-chain nerve (a quasi-category).

F.1 τ-Admissible 1-Simplices

Definition (τ-measure on chains). Assume $\mathcal{S}$ provides a function

$$\mu_\tau : N(\mathcal{S})_1 \to \mathbb{R}_{\geq 0} \cup \{\infty\}$$

assigning each 1-simplex (operator chain) a τ-instability measure.

Fix a threshold $\Lambda \in \mathbb{R}_{\geq 0}$.

Definition (τ-admissible edges). A 1-simplex $\gamma \in N(\mathcal{S})_1$ is τ-*admissible* if

$$\mu_\tau(\gamma) \leq \Lambda.$$

Let $N(\mathcal{S})_1^{(\Lambda)} \subseteq N(\mathcal{S})_1$ denote the set of τ-admissible 1-simplices.

F.2 The τ-Filtered Simplicial Subset

Definition (Filtered simplicial subset). Define a simplicial subset

$$N(\mathcal{S})^{(\Lambda)} \subseteq N(\mathcal{S})$$

by:

- $N(\mathcal{S})_0^{(\Lambda)} := N(\mathcal{S})_0$ (same objects),
- $N(\mathcal{S})_1^{(\Lambda)} := N(\mathcal{S})_1^{(\Lambda)}$ (filtered edges),
- for $n \geq 2$, include an n-simplex $\sigma \in N(\mathcal{S})_n$ iff every 1-face of σ lies in $N(\mathcal{S})_1^{(\Lambda)}$.

F.3 τ-Horn Closure

Axiom (τ-closure under inner horn filling). For every inner horn

$$h : \Lambda_k^n \to N(\mathcal{S})^{(\Lambda)} \qquad (0 < k < n),$$

there exists a filler

$$\overline{h} : \Delta^n \to N(\mathcal{S})^{(\Lambda)}.$$

Remark. This axiom states: whenever all boundary edges are τ-admissible and form a coherent partial composition, there is a τ-admissible completion.

F.4 Theorem: $N(\mathcal{S})^{(\Lambda)}$ is a Sub-∞-Category

Theorem. If $N(\mathcal{S})$ is a quasi-category and τ-closure holds, then $N(\mathcal{S})^{(\Lambda)}$ is a quasi-category. Hence it determines a sub-∞-category

$$\mathsf{U}_{\tau}^{(\Lambda)}(\mathcal{S}) \;:=\; N(\mathcal{S})^{(\Lambda)} \;\subseteq\; \mathsf{U}(\mathcal{S}).$$

Proof. A quasi-category is characterized by existence of fillers for all inner horns. By restriction, any inner horn in $N(\mathcal{S})^{(\Lambda)}$ is an inner horn in $N(\mathcal{S})$ whose 1-faces are τ-admissible. By τ-closure, it admits a filler lying in the filtered subset. Thus $N(\mathcal{S})^{(\Lambda)}$ satisfies the inner horn filler condition. □

F.5 Two Canonical Choices of μ_τ

(1) Bounded curvature. Let $\mu_\tau(\gamma)$ be the maximal curvature value attained along the chain.

(2) Collapse-survival depth. Define $\mu_\tau(\gamma) = \infty$ if γ is destroyed by collapse testing, and otherwise $\mu_\tau(\gamma)$ equals a residue score. This makes $\mathsf{U}_{\tau}^{(\Lambda)}(\mathcal{S})$ the "survivors-only" sub-∞-category.

A Foundations Appendix: Operads, Categories, and Stability in UNNS

This appendix consolidates the formal foundations of the UNNS Substrate. We present: (i) the explicit operadic signature underlying UNNS, (ii) the adjunction between UNNS substrates and ∞-categories, (iii) the role of Matrix-Mind in forcing enrichment beyond $(\infty, 1)$, and (iv) the construction of a τ-filtered observable sub-∞-category.

A.1 The Operadic Signature Σ_{UNNS}

A.1.1 Colors

Let the set of operadic colors be

$$\mathrm{Col} := \{\Phi, \Psi, \tau, 0\},$$

where 0 denotes the Zero (terminal) substrate state.

A.1.2 Generating Operations

The operadic signature

$$\Sigma_{\text{UNNS}} := (\text{Col}, \text{Gen})$$

consists of the following generators, each equipped with a typing profile

$$(c_1, \ldots, c_n) \longrightarrow c', \qquad c_i, c' \in \text{Col}.$$

Nullary.

$$\mathcal{O}_0 : () \to 0.$$

Unary phase progressors (I–IV).

$$\mathcal{O}_{\text{I}} : \Phi \to \Phi, \quad \mathcal{O}_{\text{II}} : \Phi \to \Phi, \quad \mathcal{O}_{\text{III}} : \Phi \to \Psi, \quad \mathcal{O}_{\text{IV}} : \Psi \to \Psi.$$

Octadic mid-engine with dual modes (V–VIII). For each $k \in \{\text{V}, \text{VI}, \text{VII}, \text{VIII}\}$ and each $n, m \geq 1$:

Semantic mode:

$$\mathcal{O}^{\text{sem}}_{k;n\to m} : (\Psi, \ldots, \Psi) \to (\Psi, \ldots, \Psi).$$

Structural mode:

$$\mathcal{O}^{\text{str}}_{k;n\to m} : (\tau, \ldots, \tau) \to (\tau, \ldots, \tau).$$

Unary shaping operators (IX, XI, XIV–XVII).

$$\mathcal{O}_{\text{IX}} : \tau \to \tau, \quad \mathcal{O}_{\text{XI}} : \tau \to \tau, \quad \mathcal{O}_{\text{XIV}} : \tau \to \tau, \quad \mathcal{O}_{\text{XV}} : \tau \to \tau, \quad \mathcal{O}_{\text{XVI}} : \tau \to \tau, \quad \mathcal{O}_{\text{XVII}} : \tau \to \tau.$$

Binary coupling (X, XIII).

$$\mathcal{O}_{\text{X}} : (\tau, \tau) \to \tau, \qquad \mathcal{O}_{\text{XIII}} : (\Psi, \tau) \to \tau.$$

Collapse (XII).

$$\mathcal{O}_{\text{XII}} : \tau \to 0.$$

A.1.3 Syntax vs Semantics

The signature Σ_{UNNS} specifies *only* typing and arity. A concrete UNNS substrate adds:

- admissibility predicates,
- τ-coherence bounds,
- collapse residue dynamics.

A.2 The Adjunction $\mathsf{F} \dashv \mathsf{U}$

Let Cat_∞ denote the category of quasi-categories, and let $\mathsf{UNNS}_{\text{coh}}$ be the category of τ-coherent UNNS substrates.

Forgetful functor.

$$\mathsf{U} : \mathsf{UNNS}_{\text{coh}} \to \mathsf{Cat}_\infty$$

maps a substrate $\mathcal{S}$ to its operator-chain nerve $N(\mathcal{S})$, forgetting stability and collapse semantics while retaining compositional structure.

Free functor.

$$\mathsf{F} : \mathsf{Cat}_\infty \to \mathsf{UNNS}_{\text{coh}}$$

assigns to a quasi-category X the free UNNS substrate generated by:

- objects X_0 as recursive states,
- edges X_1 as generating chains,
- higher simplices as coherence witnesses,
- the full Codex operator alphabet acting formally,
- universal collapse to Zero.

Adjunction. For all $X \in \mathsf{Cat}_\infty$ and $\mathcal{S} \in \mathsf{UNNS}_{\text{coh}}$, there is a natural bijection

$$\mathrm{Hom}_{\mathsf{UNNS}}(\mathsf{F}(X), \mathcal{S}) \;\cong\; \mathrm{Hom}_{\mathsf{Cat}_\infty}(X, \mathsf{U}(\mathcal{S})).$$

A.3 Matrix-Mind and Enrichment Beyond $(\infty, 1)$

In an $(\infty, 1)$-category, all morphisms above dimension 1 are invertible. UNNS violates this assumption.

Matrix-Mind (Operator XVII) acts not on states or chains alone, but on the *admissibility and evaluation regime* itself:

$$\mathcal{O}_{\mathrm{XVII}} : \mathcal{M} \to \mathcal{M}',$$

where $\mathcal{M}$ encodes selection criteria, scoring, and coherence thresholds.

Such updates are generally non-invertible. They induce directed transformations between admissibility structures.

Consequence. UNNS naturally supports:

- 0-cells: states or substrates,
- 1-cells: operator-chain dynamics,
- 2-cells: directed updates of evaluation/coherence regimes.

Hence the natural categorical target of full UNNS semantics is an $(\infty, 2)$-category or an $(\infty, 1)$-category enriched in directed data.

A.4 The τ-Filtered Observable Sub-∞-Category

Let $\mathcal{S} \in \mathsf{UNNS}_{\mathrm{coh}}$ and

$$\mathsf{U}(\mathcal{S}) = N(\mathcal{S})$$

its underlying quasi-category.

τ-measure. Assume a function

$$\mu_\tau : N(\mathcal{S})_1 \to \mathbb{R}_{\geq 0} \cup \{\infty\}$$

measuring instability of operator chains.

Fix a threshold Λ.

τ-filtered simplicial subset. Define $N(\mathcal{S})^{(\Lambda)}$ by:

- same 0-simplices as $N(\mathcal{S})$,
- only edges γ with $\mu_\tau(\gamma) \leq \Lambda$,
- higher simplices whose 1-faces are all τ-admissible.

τ-closure axiom. Any inner horn in $N(\mathcal{S})^{(\Lambda)}$ admits a filler lying in $N(\mathcal{S})^{(\Lambda)}$.

Theorem. Under τ-closure,

$$\mathsf{U}_\tau^{(\Lambda)}(\mathcal{S}) := N(\mathcal{S})^{(\Lambda)}$$

is a quasi-category and hence a sub-∞-category of $\mathsf{U}(\mathcal{S})$.

A.5 Interpretive Summary

- Σ_{UNNS} provides the operadic syntax.
- $\mathsf{F} \dashv \mathsf{U}$ separates syntax from compositional semantics.
- Operator XVII lifts UNNS beyond $(\infty, 1)$ by acting on coherence regimes.
- $\mathsf{U}_\tau^{(\Lambda)}(\mathcal{S})$ captures the observable, stability-selected dynamics.

This positions UNNS as a substrate in which higher-categorical structure is generated, filtered, and transformed by recursion, collapse, and evaluation.

A Appendix: Discrete Curvature κ in the UNNS Substrate

A.1 Motivation

Throughout the main text, the term *curvature* κ is used to characterize the stability and viability of recursive structures. Unlike differential geometry, the UNNS Substrate is fundamentally discrete: it operates on sequences, operator chains, and recursion depth.

This appendix provides an explicit, operational definition of κ suitable for discrete UNNS sequences and compatible with Sobra–Sobtra dynamics and Operator XII (Collapse).

A.2 Recursive Sequences and Residue

Let $\{x_n\}_{n\geq 0}$ be a discrete UNNS-generated sequence, produced by iterative application of admissible operators.

Define the *residue* r_n at step n as the component of the sequence that is rejected or suppressed by collapse-testing:

$$r_n := x_n - \text{Sobtra}(x_n),$$

where Sobtra denotes the surviving (stable) component after Sobra–Sobtra separation.

Intuitively:

- x_n is the raw recursive output,
- $\text{Sobtra}(x_n)$ is the retained structure,
- r_n measures excess or instability.

A.3 Local Discrete Curvature

Definition (Local Discrete Curvature). The local curvature at step n is defined as the normalized growth rate of the residue:

$$\kappa_n := \frac{\|r_{n+1} - r_n\|}{\|x_n\| + \varepsilon},$$

where $\|\cdot\|$ is a chosen norm on the sequence space and $\varepsilon > 0$ is a small regularization constant.

Interpretation.

- $\kappa_n \approx 0$ indicates stable recursion,
- large κ_n indicates rapidly diverging residue,
- sign or directionality (if defined) may encode phase transitions.

A.4 Integrated Curvature Along a Chain

For an operator chain (history)

$$\gamma : \ x_0 \to x_1 \to \cdots \to x_N,$$

define the *integrated curvature* as:

$$\kappa(\gamma) := \sum_{n=0}^{N-1} w_n \, \kappa_n,$$

where weights w_n may encode:

- recursion depth sensitivity,
- phase weighting (e.g. stronger penalties in τ-phase),
- operator-specific amplification factors.

This quantity is the canonical candidate for the $\mu_\tau(\gamma)$ used in τ-filtering.

A.5 Curvature and τ-Coherence

Definition (τ-Coherence via Curvature). A recursive chain γ is τ-coherent if:

$$\kappa(\gamma) \leq \Lambda,$$

for a fixed threshold Λ.

Chains exceeding this bound are considered unstable and are eventually eliminated by Operator XII.

A.6 Relation to Operator XII (Collapse)

Operator XII may be understood as the limiting case:

$$\lim_{\kappa(\gamma)\to\infty} \gamma \ \mapsto \ 0,$$

i.e. collapse absorbs chains whose curvature diverges.

Thus collapse is not an independent axiom but the *terminal response* to unbounded discrete curvature.

A.7 Alternative Curvature Metrics

The definition above is canonical but not exclusive. Other admissible curvature proxies include:

- divergence rate of Sobtra cardinality,
- variance growth across recursive shells,
- sensitivity of invariants under perturbation,
- entropy production per recursion step.

All such measures are acceptable provided they satisfy:

1. $\kappa \geq 0$,
2. κ is additive or subadditive along chains,
3. κ diverges for collapse-dominated recursion.

A.8 Summary

In the UNNS Substrate, curvature κ is:

- not geometric bending in space,
- but a discrete measure of recursive instability,
- computed from residue growth under Sobra–Sobtra separation,
- and used operationally to define τ-coherence and collapse.

This definition anchors curvature as a measurable, engine-implementable quantity, suitable for both theoretical analysis and Chamber-level experimentation.