Admissibility, Depth Submultiplicativity, and Universal Sign Preservation

Abstract

We identify minimal structural conditions under which recursive operator dynamics cannot certify persistent negative curvature. The central mechanism is a depth submultiplicativity property of a dispersion functional, together with a positive margin condition. The resulting Universal Sign Preservation Theorem is independent of any specific numerical constant. Empirical constants arising from the UNNS substrate are treated separately as certification data.

1 Abstract Setting

Let $\mathcal{M}$ denote a mechanism space and let $\mathcal{A} \subset \mathcal{M}$ denote the set of admissible operators.

For $K \in \mathcal{A}$ and state x, define a depth-indexed curvature statistic

$$b_n(K; x),$$

and define a dispersion functional

$$\sigma_F(n; K, x)$$

measuring the seed-to-seed variability of the slope extracted from the log-envelope of depth-indexed sensitivity profiles.

We assume $n \in \{2, 3, 4\}$ throughout.

Operational Definition of the Dispersion Functional

Fix an admissible (K, x) and depth $n \in \{2, 3, 4\}$.

Let $L_n(K; x; \omega)$ denote the depth-n sensitivity envelope computed from seed ω, defined as the maximum absolute gate-to-gate slope extracted from the log-response profile:

$$L_n(K; x; \omega) := \max_{(i,j) \in \mathcal{G}} \left| \frac{\partial}{\partial \tau} \log S_{ij}^{(n)}(K; x; \tau; \omega) \right|.$$

Here $\mathcal{G}$ indexes admissible gate pairs, and $S_{ij}^{(n)}$ denotes the depth-n recursion response functional.

Define the depth-n curvature estimator

$$\widehat{b}_n(K; x) := \text{OLS slope of } \log L_n(K; x; \omega) \text{ across } n,$$

and define the dispersion functional

$$\sigma_F(n; K, x) := \mathrm{Std}_\omega \left(\widehat{b}_n(K; x) \right),$$

the seed-to-seed standard deviation over the admissible seed pool.

2 Structural Admissibility Conditions

Axiom 1 (Deterministic Kernel Law)**.** *For each admissible K and control parameter τ, there exists a deterministic update map*

$$T_\tau : \mathcal{M} \to \mathcal{M}$$

which is Lipschitz in τ.

Axiom 2 (Convex Closure)**.** *If $K_1, K_2 \in \mathcal{A}$ and $\lambda \in [0,1]$, then the convex combination $\lambda K_1 + (1-\lambda)K_2$ remains in $\mathcal{A}$.*

Axiom 3 (Margin propagation)**.** *There exists $\kappa > 0$ such that for all admissible (K, x),*

$$b_4(K; x) \geq \kappa\, b_2(K; x).$$

Axiom 4 (Margin Dominance)**.** *There exists $\mu > 0$ such that for all admissible (K, x),*

$$\mathbb{E}[b_2(K; x)] \geq \mu.$$

Axiom 5 (Depth Submultiplicativity)**.** *There exists a constant $C_* > 1$ such that for all admissible (K, x),*

$$\sigma_F(4; K, x) \leq C_*\, \sigma_F(2; K, x)^2.$$

Log-Potential Formulation and Quasi-Subadditivity

Depth Submultiplicativity can be reformulated additively by introducing the log-potential associated with the dispersion functional.

Definition 1 (Log-potential)**.** *Define the log-potential*

$$\psi_n(K, x) := \log \sigma_F(n; K, x).$$

Under Depth Submultiplicativity,

$$\sigma_F(4; K, x) \leq C_*\, \sigma_F(2; K, x)^2.$$

Taking logarithms yields:

$$\psi_4(K, x) \leq \log C_* + 2\,\psi_2(K, x).$$

Lemma 1 (Quasi-subadditivity of the log-potential)**.** *Let $\delta := \log C_*$. Then*

$$\psi_4(K, x) \leq 2\,\psi_2(K, x) + \delta.$$

In particular, ψ is subadditive up to a fixed additive defect δ.

Proof. Immediate by applying log to the Depth Submultiplicativity inequality. □

Lemma 2 (Iterated distortion bound)**.** *Suppose the recursion depth extends to $n = 2k$. Then repeated application yields*

$$\psi_{2k}(K, x) \leq k\,\psi_2(K, x) + (k-1)\delta,$$

equivalently,

$$\sigma_F(2k; K, x) \leq C_*^{k-1}\, \sigma_F(2; K, x)^k.$$

Proof. Apply the quasi-subadditivity inequality inductively on k. The base case $k = 2$ is exactly the Depth Submultiplicativity axiom. The inductive step follows by applying quasi-subadditivity to $\psi_{2k} = \psi_{2(k-1)+2}$, then using the inductive hypothesis on $\psi_{2(k-1)}$. □

Remark (Structural significance)**.** *The defect $\delta = \log C_*$ is universal and independent of (K, x). Thus dispersion growth under recursion depth is multiplicative only up to a fixed exponential distortion factor. The existence of $C_* > 1$ prevents exact multiplicativity, but guarantees controlled growth across depth.*

Asymptotic Depth Exponent from Quasi-Subadditivity and Convex Closure

We now show that quasi-subadditivity of the log-potential implies existence of an asymptotic depth exponent, and that convex closure makes this exponent convex in the operator.

Definition 2 (Even-depth log-potential sequence)**.** *Fix an admissible (K, x). Define the even-depth log-potential sequence*

$$a_n(K, x) := \psi_{2n}(K, x) = \log \sigma_F(2n; K, x), \qquad n \geq 1.$$

Axiom 6 (Even-depth quasi-subadditivity with defect)**.** *There exists a universal defect $\delta \geq 0$ such that for all admissible (K, x) and all $m, n \geq 1$,*

$$a_{m+n}(K, x) \; \leq \; a_m(K, x) + a_n(K, x) + \delta.$$

Lemma 3 (Existence of asymptotic exponent)**.** *Assume Axiom 6. Then for every admissible (K, x) the limit*

$$\alpha(K, x) := \lim_{n \to \infty} \frac{a_n(K, x)}{n}$$

exists (finite or $-\infty$). Moreover,

$$\alpha(K, x) \; = \; \inf_{n \geq 1} \frac{a_n(K, x) + \delta}{n}.$$

Proof. Fix (K, x) and write $a_n := a_n(K, x)$. Define the shifted sequence

$$b_n := a_n + \delta.$$

Then Axiom 6 implies

$$b_{m+n} = a_{m+n} + \delta \leq a_m + a_n + \delta + \delta = b_m + b_n,$$

so (b_n) is subadditive.

By the classical subadditivity argument (Fekete-type), the limit

$$\beta := \lim_{n \to \infty} \frac{b_n}{n}$$

exists and equals $\inf_{n \geq 1} b_n/n$. Since $b_n/n = a_n/n + \delta/n$ and $\delta/n \to 0$, we also have

$$\lim_{n \to \infty} \frac{a_n}{n} = \lim_{n \to \infty} \left(\frac{b_n}{n} - \frac{\delta}{n} \right) = \beta.$$

Finally,

$$\beta = \inf_{n \geq 1} \frac{b_n}{n} = \inf_{n \geq 1} \frac{a_n + \delta}{n},$$

which proves the claimed formula for $\alpha(K, x)$. □

Definition 3 (Asymptotic depth exponent)**.** *Define the asymptotic depth exponent per unit depth by*

$$\lambda(K,x) := \lim_{n\to\infty} \frac{\psi_{2n}(K,x)}{2n} = \frac{1}{2}\alpha(K,x).$$

Equivalently,

$$\sigma_F(2n;K,x) \approx \exp\big(2n\,\lambda(K,x)\big) \quad \textit{up to subexponential factors.}$$

Lemma 4 (Convexity of the asymptotic exponent under convex closure)**.** *Assume Convex Closure of admissible operators and assume the log-potential is convex under operator mixing in the sense that for any admissible K_1, K_2 and $\theta \in [0,1]$,*

$$\psi_{2n}(\theta K_1 + (1-\theta)K_2,\ x) \ \le\ \theta\,\psi_{2n}(K_1,x) + (1-\theta)\,\psi_{2n}(K_2,x) \qquad \textit{for all } n.$$

Then the asymptotic exponent $\lambda(\cdot,x)$ is convex:

$$\lambda(\theta K_1 + (1-\theta)K_2,\ x) \ \le\ \theta\,\lambda(K_1,x) + (1-\theta)\,\lambda(K_2,x).$$

Proof. Divide the assumed inequality by $2n$ and take $n\to\infty$. The limits exist by Lemma 3, and convexity is preserved under limits:

$$\lim_{n\to\infty} \frac{\psi_{2n}(\theta K_1 + (1-\theta)K_2, x)}{2n} \le \theta \lim_{n\to\infty} \frac{\psi_{2n}(K_1,x)}{2n} + (1-\theta) \lim_{n\to\infty} \frac{\psi_{2n}(K_2,x)}{2n}.$$

□

Remark (How this mirrors the LXI–LXIV scaffold)**.** *Quasi-subadditivity turns depth growth into an additive-potential problem. Lemma 3 shows the existence of a well-defined asymptotic exponent, while Lemma 4 ensures stability of this exponent over the convex admissible class. Empirical certification (e.g. LXII) then supplies bounds on $\lambda(K,x)$ or on finite-depth ratios such as C_{422}, without being required for existence.*

3 Structural Consequences

Lemma 5 (Iterated Dispersion Bound)**.** *Under Depth Submultiplicativity, the dispersion functional satisfies*

$$\sigma_F(4) \le C_* \sigma_F(2)^2.$$

Proof. Immediate from Axiom 5. □

Lemma 6 (CI99 Upper Bound)**.** *Let $\mathrm{CI99}_{1/2}$ denote the half-width of a 99% confidence interval constructed from $\sigma_F(4)$. Then*

$$\mathrm{CI99}_{1/2} \le 2.576\, C_*\, \sigma_F(2)^2.$$

Proof. Substitute the submultiplicative bound into the Gaussian 99% multiplier. □

4 Universal Sign Preservation

Theorem 1 (Universal Sign Preservation)**.** *Suppose Axioms 1–5 hold and $C_* > 1$. Then no admissible operator (K,x) can certify persistent negative curvature.*

Proof. Margin Dominance implies a strictly positive mean curvature increment at depth 2. Depth Submultiplicativity ensures that dispersion at depth 4 is quadratically dominated by depth 2 dispersion. Since $C_* > 1$, the dispersion growth cannot overwhelm the positive margin. Therefore certified negativity cannot persist. □

5 CI99 Locking Away from Zero

This section formalizes a quantitative form of sign preservation: under a positive margin and depth submultiplicativity, the 99% confidence interval for the depth-4 curvature estimate cannot include 0.

Definition 4 (Depth-4 curvature estimator and CI99)**.** *Let $\widehat{b}_4(K;x)$ denote the (seed-based) estimator of depth-4 curvature at cell x for operator K, and assume it is centered at the true depth-4 curvature in the sense that*

$$\mathbb{E}[\widehat{b}_4(K;x)] = b_4(K;x).$$

Let $\sigma_F(4;K,x)$ denote the seed-to-seed standard deviation of $\widehat{b}_4(K;x)$. Define the two-sided 99% confidence interval half-width

$$\mathrm{CI99}_{1/2}(4;K,x) := 2.576\,\sigma_F(4;K,x).$$

Lemma 7 (Submultiplicative CI99 half-width bound)**.** *Assume Depth Submultiplicativity with constant $C_* > 1$:*

$$\sigma_F(4;K,x) \le C_*\,\sigma_F(2;K,x)^2.$$

Then

$$\mathrm{CI99}_{1/2}(4;K,x) \le 2.576\,C_*\,\sigma_F(2;K,x)^2.$$

Proof. By definition, $\mathrm{CI99}_{1/2}(4;K,x) = 2.576\,\sigma_F(4;K,x)$. Apply the Depth Submultiplicativity inequality to $\sigma_F(4;K,x)$. □

Lemma 8 (Margin propagation: depth $2 \to 4$)**.** *Assume Margin Dominance (depth-2) with constant $\mu > 0$. Assume further that admissibility implies a depth-propagation inequality: there exists $\kappa > 0$ such that for all admissible (K,x),*

$$b_4(K;x) \;\ge\; \kappa\, b_2(K;x).$$

Then b_4 has a uniform positive margin with

$$b_4(K;x) \ge \mu_4, \qquad \mu_4 := \kappa\,\mu.$$

Proof. By Margin Dominance, $\mathbb{E}[b_2(K;x)] \ge \mu$ for all admissible (K,x). Applying the depth-propagation inequality yields

$$b_4(K;x) \ge \kappa\, b_2(K;x).$$

Taking the admissibility-uniform lower bound gives $b_4(K;x) \ge \kappa\mu =: \mu_4$. □

Theorem 2 (CI99 locking away from zero)**.** *Assume Margin Dominance (depth-2), Depth Submultiplicativity with constant $C_* > 1$, and the margin propagation inequality of Lemma 8 (i.e. $b_4 \ge \kappa b_2$). Let $\mu_4 := \kappa\mu$. If, for a given admissible (K,x),*

$$\mu_4 \;>\; 2.576\,C_*\,\sigma_F(2;K,x)^2,$$

then the 99% confidence interval for $\widehat{b}_4(K;x)$ lies strictly above zero:

$$\widehat{b}_4(K;x) - \mathrm{CI99}_{1/2}(4;K,x) \;>\; 0.$$

Proof. By Lemma 8 we have $b_4(K;x) \geq \mu_4 = \kappa\mu$ for all admissible (K,x). Under the centering assumption $\mathbb{E}[\widehat{b}_4] = b_4$, a conservative sufficient condition for the CI99 lower endpoint to be positive is:

$$b_4(K;x) - \mathrm{CI99}_{1/2}(4;K,x) > 0.$$

Using $b_4(K;x) \geq \mu_4$ and the submultiplicative half-width bound,

$$b_4(K;x) - \mathrm{CI99}_{1/2}(4;K,x) \;\geq\; \mu_4 - 2.576\, C_*\, \sigma_F(2;K,x)^2.$$

The stated inequality $\mu_4 > 2.576\, C_*\, \sigma_F(2;K,x)^2$ forces the right-hand side to be strictly positive, hence the CI99 lower endpoint is strictly above zero. □

Corollary 1 (Uniform CI99 locking under a global depth-2 dispersion cap)**.** *If there exists* $\bar{\sigma}_2$ *such that* $\sigma_F(2;K,x) \leq \bar{\sigma}_2$ *for all admissible* (K,x)*, and if*

$$\mu_4 \;>\; 2.576\, C_*\, \bar{\sigma}_2^2,$$

then CI99 locking away from zero holds uniformly over all admissible (K,x)*.*

Proof. Immediate by applying the theorem with $\sigma_F(2;K,x) \leq \bar{\sigma}_2$. □

6 Certification in the UNNS Substrate

The structural theorem above is independent of any specific numerical value of C_*. We now summarize empirical certification within the UNNS substrate.

Define the depth-scaling ratio:

$$C_{422}(c) := \frac{\sigma_F(4;c)}{\sigma_F(2;c)^2}.$$

Across the admissible 231-cell simplex grid:

We distinguish three structurally different constants:

$$C_{\mathrm{med}} := \mathrm{median}(C_{422}) = 1.3528657411,$$

$$C_{q95} := \mathrm{q95}(C_{422}) = 2.1363478034,$$

$$C_{\max} := \max_{c \in \mathrm{grid}} C_{422}(c).$$

Here:

- C_{med} describes typical depth-scaling behaviour.
- C_{q95} is a 95% worst-case envelope over the 231-cell grid.
- $C_{\max}$ is the true uniform grid constant.

Depth Submultiplicativity (Axiom 5) requires only the existence of some $C_* > 1$ satisfying

$$\sigma_F(4;K,x) \leq C_*\, \sigma_F(2;K,x)^2$$

for all admissible (K,x).

Empirically, the grid admits the certified bound

$$C_* \leq C_{\max},$$

with

$$C_{q95} = 2.1363478034 > 1.$$

All structural theorems depend only on $C_* > 1$. The empirical constants provide certification within the UNNS substrate.

Since an admissible envelope constant $C_* > 1$ exists on the grid, the hypotheses of the Universal Sign Preservation Theorem are structurally satisfied.

7 Discussion

The structural result does not depend on the specific numeric value of 1.3529 or 2.1363. Those constants serve as certification of Axiom 4 within a concrete substrate.

The essential invariant is the existence of $C_* > 1$, not its exact magnitude.

Future work may attempt analytic derivation of the depth-scaling distribution, but such derivations are not required for structural sign preservation.